GROUPS AND DYNAMICS

ARISTOTELIS PANAGIOTOPOULOS

Introduction

This is a topics course in topological groups and dynamics. We will be focusing on techniques and phenomena that go beyond the realm of locally compact groups. In the absence of classical techniques which rely on local compactness, we will develop various frameworks for analyzing the structure of "large" topological groups, such as: Fraïssé theoretic methods; Baire category methods; structured completions and compactifications. As we develop these techniques we will provide a wide range of applications and examples from topology, analysis, and logic. Here are three centerpieces that, among others, will be covered:

- **Extreme amenability**. Consider the automorphism group $\mathrm{Aut}(\mathbb{Q},<)$, of all order preserving bijections of the rationals. Endowed with the pointwise convergence topology it has the astonishing property that whenever it acts continuously on a compact space K, K has a fixed point under the action. As we will see, this is intimately tied to a combinatorial principle known as the Ramsey theorem. We will develop the theory of a more general phenomenon known as the KPT-correspondence (for Kechris, Pestov, Todorcevic).
- **Anti-classification results**. Dynamics play a very important role in logic because they allow us to prove negative results about classification projects in mathematics. We will see for example that there is no "constructive way" of classifying countable graphs up to isomorphism by assigning to them enough meaningful invariants. By developing more intricate tools (such as Hjorth's turbulence theory) we will see that similar classification projects from analysis and topology are even more hopeless (e.g. classifying unitary operators up to unitary equivalence).
- **Exotic groups.** Consider the group $\mathrm{Homeo}_+([0,1])$, of all orientation preserving homeomorphisms of the compact space $[0,1]$. Endowed with the compact-open topology it has no non-trivial continuous representations on a Hilbert space. Even more, by a recent result of Megrelishvili, it has no non-trivial representation on any reflexive Banach space. As we will see, this is intimately tied to the the fact that the group $\mathrm{Homeo}_+([0,1])$ has no compactification on which the multiplication of $\mathrm{Homeo}_+([0,1])$ extends to a seperately continuous semigroup operation.

1. Week 1

A **Polish space** is a topological space X that is separable and completely metrizable, i.e., it admits a complete metric d that induces the topology on X. Here are some facts that we are going to use regarding Polish spaces. Let X be a Polish space. A set $A \subseteq X$ is **nowhere dense** if for every non-empty open U in X, the set $U \setminus \overline{A}$ is non-empty. A subset $M \subseteq X$ is **meager** if M is a countable union of nowhere dense sets. A subset $C \subseteq X$ is **co-meager** if C^c is meager.

Fact 1. *Let X be a Polish space.*

(1) If Z is a subspace of X (with the subspace topology) then Z is Polish if and only if Z is G_δ in X.
(2) **Baire category theorem.** *If (C_n) is a sequence of dense and open sets in X then $\cap_n C_n$ is dense.*
(3) every comeager set C contains a dense G_δ.

A **topological group** is a group G endowed with a topology which makes the operations $(g,h) \mapsto g \cdot h$, $g \mapsto g^{-1}$ continuous.

Check. One can equivalently define topological groups as those groups endowed with a topology which makes $(g,h) \mapsto g \cdot h^{-1}$ continuous.

Definition 2. A **Polish group** is a topological group whose topology is Polish.

Countable discrete groups Γ such as $(\mathbb{Z},+), (\mathbb{F}_2,\cdot)$, etc., or more generally, locally compact metrizable groups such as $(\mathbb{R},+)$, $SL_2(\mathbb{R})$, etc., are Polish.

However, our treatment will include Polish groups which are far from being locally compact. Here are some families of Polish groups which we will consider.

(1) Permutation groups. Consider S_∞ to be the group consisting of all bijections from $\mathbb{N}$ to $\mathbb{N}$ where the group operation is composition $(gh)(n) = g(h(n))$. The topology on S_∞ is the **pointwise convergence** topology, i.e., the smallest topology with the property that (g_k) converges to g, if and only if, for all $n\mathbb{N}$ we have that

$$\lim_k g_k(n) = g(k), \quad \text{or equivalently,} \quad \exists k_0 \, \forall k > k_0 \, g_k(n) = g(n).$$

In other words, the collection of all sets of the form:

$$V_{A,f} \colon = \{g \in S_\infty \mid f{\restriction}A = g{\restriction}A\},$$

where A varies over all finite subsets of $\mathbb{N}$ and f varies over elements of S_∞, constitutes a basis for the topology on S_∞.

To see that S_∞ is a Polish group notice that the subgroup S_∞^{fin} of all permutations with finite support is countable and dense in S_∞ and that S_∞ is a G_δ subset of the Baire space $\mathbb{N}^\mathbb{N}$ of all functions from $\mathbb{N}$ to $\mathbb{N}$ endowed also with the pointwise convergence topology.

Alternatively one can directly define a complete metric for S_∞ as follows. First consider the metric $d(g,h) = 2^{-n}$ iff $g \neq h$ and n is the smallest natural number so that $g(n) \neq h(n)$. Let also $D(g,h) = d(g,h) + d(g^{-1}, h^{-1})$. Then D is a complete metric on S_∞ compatible with the topology.

Check. The metric d is compatible with the topology (and it is left invariant) but it is **not** complete! (d keeps track of where n maps to, but not what is mapped on n)

We will be interested in closed subgroups of S_∞. As an example consider the group $\mathrm{Aut}(\mathbb{Q}, \leq)$, of all order-preserving bijections from $\mathbb{Q}$ to $\mathbb{Q}$. By fixing an enumeration of $\mathbb{Q}$ we can view $\mathrm{Aut}(\mathbb{Q}, \leq)$ as a subgroup of S_∞, which is moreover closed, since not being in $\mathrm{Aut}(\mathbb{Q}, \leq)$ can be detected by the fact that a finite $A = \{a \leq b\} \subset \mathbb{Q}$ is mapped to $f(a) \not\leq f(b)$. In fact, we will see that all closed subgroups of S_∞ are of the form $\mathrm{Aut}(M)$, for some appropriate $\mathcal{L}$-structure M on domain $\mathbb{N}$.

(2) Isometry groups. Let (X, ρ) be a Polish metric space and consider the group $\mathrm{Iso}(X, \rho)$ of all isometries of X with multiplication being the composition. An **isometry** of X is any bijection g from X to X with $\rho(x, x') = \rho(g(x), g(x'))$. As in (1), we view $\mathrm{Iso}(X, \rho)$ with the pointwise convergence topology, i.e., the collection of all sets of the form:

$$V_{A,f,\varepsilon} \colon = \{g \in \mathrm{Iso}(X, \rho) \mid \text{ for all } x \in A \ \rho(g(x), f(x)) < \varepsilon\},$$

where A varies over all finite subsets of $\mathrm{Iso}(X, \rho)$, f varies over elements of $\mathrm{Iso}(X, \rho)$, and ε is any positive real, constitutes a basis for the topology on $\mathrm{Iso}(X, \rho)$. It is easy to see that $\mathrm{Iso}(X, \rho)$ is a G_δ subset of the Polish space $\mathrm{Emb}(X, \rho)$ of all isometric embeddings from X to X endowed with the pointwise convergence topology. It is therefore Polish. We can build a left invariant metric d on $\mathrm{Iso}(X, \rho)$ by fixing some dense sequence (x_n) in X and setting

$$d(g, f) = \sum_n \frac{1}{2^n} \frac{\rho(f(x_n), g(x_n))}{1 + \rho(f(x_n), g(x_n))}$$

As in (1), one can check that while the metric d is not complete in general, the metric D given by $D(f, g) = d(f, g) + d(f^{-1}, g^{-1})$ is a complete and compatible metric on $\mathrm{Iso}(X, \rho)$. In fact, a corollary of Theorem 8 implies that D is complete, given that d is left invariant.

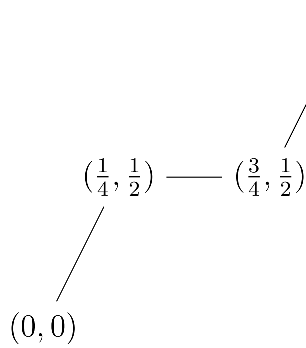
There are many Polish groups of interest, which are closed subgroups of $\mathrm{Iso}(X, \rho)$, for various (X, ρ). One example is the unitary group $\mathcal{U}(\mathcal{H})$ of any separable Hilbert space $\mathcal{H}$. Moreover, notice that the examples under (1) are in fact examples of the form (2), since S_∞ is of the form $\mathrm{Iso}(X, \rho)$, where X is $\mathbb{N}$ and ρ is the discrete $\{0, 1\}$-valued metric.

(3) Homeomorphism groups. Let X be any compact metrizable topological space and let $\mathrm{Homeo}(X)$ be the group of all continuous bijections from X to X. We view $\mathrm{Homeo}(X)$ as a topological group endowed with the compact-open topology. Since $\mathrm{Homeo}(X)$ is a G_δ subset of the Polish space $C(X, X)$ it is Polish. To find an explicit complete metric we can again invoke Theorem 8 after we provide a left invariant metric for $\mathrm{Homeo}(X)$. Let ρ be any compatible metric on X and let d_u be the uniform metric on Homeo(X) with respect to ρ, i.e.,

$$d_u(g,f) = \sup\{\rho(f(x), g(x)) \mid x \in X\}.$$

It is easy to see that $d_u(g,f)$ is a compatible right invariant metric. Hence $d(f,g) = d_u(f^{-1}, g^{-1})$ is a compatible left invariant metric. The usual, by now, recipe gives as a complete metric D that is compatible with the topology.

Exercise. Show that ρ is not complete, in general, by constructing a sequence (g_n) of elements of Homeo(X) which converge to the function whose graph is



(4) Banach spaces. Any separable Banach space is an abelian Polish group with multiplication being vector addition. Important examples will be the Banach spaces $(l_p, +)$, $p \in [1, \infty)$, of all p-summable sequences.

When it comes to Polish groups the extra uniform structure gives a strengthening of Fact 1 (1).

Lemma 3. *Let G be a Polish group and let H be a Polish subgroup of G. Then H is closed.*

Proof. By 1(1) H is G_δ and dense in the Polish group $\overline{H}$. If $g \in \overline{H} \setminus H$ then gH would also be dense G_δ in $\overline{H}$ but since it would be a different coset of H we would have $H \cap gH = \emptyset$, contradicting Fact1(2). □

The rigidity resulting from the combination of the Baire category theorem and the uniform structure of a Polish group is best exemplified by Banach-Pettis lemma and its corollaries. To motivate this result consider the following question:

Question 4. Let End$((\mathbb{R}, +))$ be the collection of all homomorphisms $\pi\colon (\mathbb{R}, +) \to (\mathbb{R}, +)$, of the additive group $(\mathbb{R}, +)$.

(1) Classify End$((\mathbb{R}, +))$;
(2) Classify the definable (e.g., Borel) members of End$((\mathbb{R}, +))$;
(3) Classify the continuous members of End$((\mathbb{R}, +))$;

For the first problem, we can view $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and by axiom of choice fix a Hamel basis Hamel$_{\mathbb{Q}}(\mathbb{R})$. End$((\mathbb{R}, +))$ can be now seen to be in bijective correspondence with the collection of all functions from Hamel$_{\mathbb{Q}}(\mathbb{R})$ to $\mathbb{R}$. The last problem has also a "simple" solution. Let π be a continuous homomorphism. Continuity says that π is determined by $\pi\restriction\mathbb{Q}$ and the fact that π is a homomorphism says that $\pi\restriction\mathbb{Q}$ is completely determined by $\pi\restriction\{1\}$. The continuous members of $\mathrm{End}((\mathbb{R},+))$ are in bijective correspondence with the collection of all functions from $\{1\}$ to $\mathbb{R}$,i.e., with $\mathbb{R}$. While the second problem looks difficult, at a first glance, Banach-Pettis lemma reduces it to the third problem which we have already solved.

Recall the collection of Borel sets in a topological space X is the smallest σ-algebra $\mathcal{B}(X)$ containing the open sets of X and the collection of Baire measurable sets is the smallest σ-algebra $\mathcal{BP}(X)$ containing the open sets and the meager sets of X. Alternatively, $\mathcal{BP}(X)$ consists of all sets of the form $M\triangle U = (M/U)\cup(U/M)$, where U is open and M is meager.

Definition 5. Let X be a Polish space and let A be any subset of X. We will denote by $U(A)$ the largest open set in which A is comeager.

Before we proceed we give a brief justification that this definition makes sense, i.e., that such $U(A)$ always exists. To see this, let $\mathcal{O}(A)$ be the collection of all open sets in which A is comeager and set $U(A) := \bigcup\{U \mid U \in \mathcal{O}(A)\}$. It follows that $U(A)$ is open and it overlays every open set in which A is comeager. So we are left to show that A is comeager in $U(A)$. This follows by the Baire category theorem since by separability of X we can find a countable collection (U_n) of elements from $\mathcal{O}(A)$ so that $\bigcup_n U_n = \bigcup\{U \mid U \in \mathcal{O}(A)\}$.

Lemma 6 (Banach-Pettis)**.** *Let G be a Polish group and let $A, B \in \mathcal{BP}(G)$. Then*

$$U(A)U(B) \subseteq AB.$$

Proof. If x is in $U(A)U(B)$, then the set $xU(B)^{-1}\cap U(A)$ is open and non-empty, since $x = a\cdot b \implies x\cdot b^{-1} = a$. But $xU(B)^{-1}\cap U(A) = U(xB^{-1})U(A)$, since inversion and translation are homeomorphisms of G.

Hence both xB^{-1} and A are comeager in the non-empty open set $U(xB^{-1})U(A)$ and therefore $xB^{-1}\cap A \neq \emptyset$, i.e., $x \in AB$. □

Theorem 7. *If $\pi\colon G\to H$ is a Baire-measurable homomorphism between Polish groups then π is continuous.*

Proof. It suffice to show that π is continuous at 1_G. Let therefore V be an open neighborhood of $1_H = \pi(1_G)$ in H and notice that by continuity of $(x, y)\mapsto x\cdot y$ and $x\mapsto x^{-1}$ we can find an open $W\subseteq V$ in H, with $1_H\in W$ and $WW^{-1}\subseteq V$.

But then, $\pi^{-1}(W)$ is Baire-measurable and non-meager, since countably many translates of $\pi^{-1}(W)$ cover G (because the same is true for W and H). By Banach-Pettis lemma we have:

$$1_G \in U(\pi^{-1}(W))U(\pi^{-1}(W))^{-1} \subseteq \pi^{-1}(W)\pi^{-1}(W)^{-1} \subseteq \pi^{-1}(V).$$

□

In the sequel we will see Polish groups which satisfy the much stronger property that arbitrary homomorphisms mapping out of them are continuous. In other words, their algebra dictates their topology. A Polish group like this is S_∞. However not every Polish group has this property.

Exercise. Show that the abstract group $(\mathbb{R},+)$ admits many different Polish topologies, i.e., there are two Polish groups which are not continuously isomorphic but if you forget the topology, both are isomorphic to the group $(\mathbb{R},+)$. Show actually that $(\mathbb{R},+)$ admits infinitely many different Polish topologies.

Exercise. (Solecki)

(1) Let (H_n) be a collection of subgroups of the Polish group G. Assume that H_n has the Baire-property, for all n, and that $G \setminus \bigcup_n H_n$ is meager. Show that there is n so that G/H_n is countable.
(2) the group $\bigoplus_n (\mathbb{Z}_{2^n})^{\mathbb{N}}$ does not carry a Polish group topology.

Project 1. There are many topological groups which do not admit any Polish topology. Moreover the arguments involved in showing that such a group is not Polish can be quite intricate. Collect examples of such groups and simplify/conceptualize existing proofs (see Shelah, Mann, Solecki, Rosendal, ...)

As we pointed out in the examples, for a Polish group G, in order to find a compatible complete metric it suffice to find a left invariant metric.

Theorem 8. *Let G be a topological group endowed with a compatible metric d that is left invariant. Let also $D(g,h) := d(g,h)+d(g^{-1},h^{-1})$. Then D is also a compatible metric for G, and if $(\widehat{G},\widehat{D})$ is the completion of (G,D) then the multiplication of G extends uniquely to $\widehat{G}$, so that $\widehat{G}$ becomes a topological group, in the topology generated by $\widehat{D}$.*

Before we prove Theorem 8 we observe that together with Lemma 3 it implies the following corollary.

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Corollary 9. *If G is a Polish group and d is a compatible left invariant metric on G then $D(g,h) := d(g,h) + d(g^{-1},h^{-1})$ is a compatible complete metric on G.*

Proof of Theorem 8. Uniqueness of multiplication follows from the fact that G is dense in $\widehat{G}$. So it suffice to show existence. For that we first record the following claim and we show how existence follows from this claim. Then we prove the claim

Claim. *If $\hat{x},\hat{y} \in \widehat{G}$ and $\varepsilon > 0$ then there exists a $\delta > 0$ so that if $x,x' \in X$ are δ-close to $\hat{x}$ (with respect to $\widehat{D}$) and y,y' are δ-close to $\hat{y}$ (with respect to $\widehat{D}$) then $\widehat{D}(xy,x'y') < \varepsilon$.*

Given this claim the rest follows easily. Since if $x_n \to \hat{x}$ and $y_n \to \hat{y}$ then the claim shows that $(x_n y_n)$ is Cauchy and therefore it converges to some $\hat{z} \in \widehat{G}$. Moreover by the claim, $\hat{z}$ will not depend on the choice of $(x_n),(y_n)$.

This defines a continuous (by the claim) multiplication on $\widehat{G}$ which extends the one on G, it is associative, and 1 is the identity. For inversion, notice that $x \mapsto x^{-1}$

is a D-isometry and therefore it uniquely extends to a $\widehat{D}$-isometry $\hat{x} \mapsto \hat{x}^*$. Using the fact that multiplication is continuous on $\widehat{G}$ it follows that $\hat{x} \cdot \hat{x}^* = \hat{x}^* \cdot \hat{x} = 1$. So, we are left with proving the claim.

Proof of Claim. First observation that by triangle inequality and left invariance of d we have that $d(gh, 1) \leq d(g, 1) + d(h, 1)$ for all $g, h \in G$. If now x, x', x_0 and y, y', y_0 are in G, by left invariance of d and the above observation we have that:

$$D(xy, x'y') = d(xy, x'y') + d(y^{-1}x^{-1}, (y')^{-1}(x')^{-1}) =$$

$$= d(1, y'x'x^{-1}y^{-1}) + d(x'y'y^{-1}x^{-1}, 1) \leq$$

$$\leq \big(d(1, y'{y_0}^{-1}) + d(1, y_0x'x^{-1}{y_0}^{-1}) + d(1, y_0y^{-1})\big) +$$

$$+\big(d(x'{x_0}^{-1}, 1) + d(x_0y'y^{-1}{x_0}^{-1}, 1) + d(x_0x^{-1}, 1)\big)$$

Fix now some x_0 and some y_0 which are $\varepsilon/10$=close to $\hat{x}$ and $\hat{y}$ respectively.

Notice now that if we chose δ to be any number less than $\varepsilon/10$ and x, x', y, y' are chosen to be δ-close to $\hat{x}, \hat{y}$ respectively then each of the terms $d(1, y'{y_0}^{-1})$, $d(1, y_0y^{-1})$, $d(x'{x_0}^{-1}, 1)$ and $d(x_0x^{-1}, 1)$ is less than $\delta + \varepsilon/10 < \varepsilon/5$.

So, by restricting δ to be perhaps even smaller, we would like for these fixed x_0, y_0 to make each of the remaining terms $d(1, y_0x'x^{-1}{y_0}^{-1})$, $d(x_0y'y^{-1}{x_0}^{-1}, 1)$ to be less than $\varepsilon/10$, as well. But notice that for these fixed x_0, y_0 the maps $z \mapsto x_0z{x_0}^{-1}$ and $z \mapsto y_0z{y_0}^{-1}$ are continuous and send 1 to 1. Let therefore δ be small enough so that additionally $x_0B(1, 2\delta){x_0}^{-1} \subseteq B(1\varepsilon/10)$ and $y_0B(1, 2\delta){y_0}^{-1} \subseteq B(1, \varepsilon/10)$, and notice that this choice finishes the proof. □

□

2. Week 2

Theorem 10 (Birkhoff-Kakutani)**.** *Let G be a topological group. Then G is Hausdorff and admits a countable neighborhood basis at 1 if and only if it admits a compatible metric d which can be always taken to be left invariant.*

Proof. We show the only non-trivial direction ⇒. Let (U_n) be any countable neighborhood basis at 1 so that $U_0 = G$. We construct a sequence of open neighborhoods (V_n) of 1 so that

(1) $V_0 = U_0 = G$ and $V_{n+1} \subseteq V_n$;
(2) $V_n = {V_n}^{-1}$;
(3) $V_n \cdot V_n \cdot V_n \subseteq V_{n+1}$;
(4) $V_n \subseteq U_n$.

This is an easy task given the continuity of $x \mapsto x^{-1}$ and $x \mapsto x \cdot x \cdot x$. Given this (V_n), for every $x, y \in G$ we define

$$\delta(x, y) = \inf\{2^{-n} \mid y^{-1}x \in V_n\}.$$

Notice that δ is well defined due to (1), left invariant, $\delta(x,x)=0$, $x \neq y \implies \delta(x,y)$, and $\delta(x,y)=\delta(y,x)$ due to (2). The only thing δ fails, in general, to satisfy is the triangle inequality. We remedy this by defining

$$d(x,y)=\inf\{\sum_{i=0}^{n-1}\delta(x_i,x_{i+1}) \mid x_0=x, x_n=y, x_i\in G, n\geq 1\}.$$

In the rest we show that d is a left invariant metric that is compatible with the topology.

It is clear that d satisfies triangle inequality and that d inherits from δ left invariance, symmetry and the property $d(x,x)=0$. So in order to show that d is a metric we are left with showing that $d(x,y)=0 \implies x=y$.

Claim. *For every $x,y\in G$ we have that $d(x,y)\geq \frac{1}{2}\delta(x,y)$.*

Proof of Claim. We will prove by induction on $n\geq 1$ that for all $x_0,\ldots,x_n$,

$$(*) \qquad \sum_{i=0}^{n-1}\delta(x_i,x_{i+1}) \geq \frac{1}{2}\delta(x_0,x_n).$$

For $n=1$ is is clear. Assume now inductively that $(*)$ holds for all m with $m<n$.

We will need to use the following property of δ which follows by (3) above: if $\delta(p,q),\delta(q,r),\delta(r,s)\leq m$ then $\delta(p,s)\leq 2m$.

Let S be the sum in the left hand side of $(*)$. If $\delta(x_0,x_1)$ is "too large", i.e, $\geq \frac{1}{2}S$ then $\sum_{i=1}^{n-1}\delta(x_i,x_{i+1})$ is "too small", i.e., $\leq \frac{1}{2}S$ and therefore, by inductive hypothesis $\delta(x_1,x_n)\leq 2\frac{1}{2}S=S$. So, by the above property of δ we have that $\delta(x_0,x_n)\geq \delta(x_0,x_1)+\delta(x_1,x_n)\leq 2S$. The same argument works if $\delta(x_{n-1},x_n)$ is "too large". We can therefore assume that $n\geq 3$ and that both $\delta(x_0,x_1)$ and $\delta(x_{n-1},x_n)$ are $<\frac{1}{2}S$. This implies that the largest r with the property that $\sum_{i=0}^{r-1}\delta(x_i,x_{i+i})$ is "too small", i.e., $\leq \frac{1}{2}S$, satisfies $1\leq r\leq n-2$. By induction hypothesis we have that

$$\delta(x_0,x_r)\leq 2\sum_{i=0}^{r-1}\delta(x_i,x_{i+i})\leq S.$$

Again by induction, and since $\sum_{i=0}^{r}\delta(x_i,x_{i+i})$ is "too large", we have that

$$\delta(x_{r+1},x_n)\leq 2\sum_{i=r+1}^{n-1}\delta(x_i,x_{i+i})\leq S.$$

Trivially we also have that $\delta(x_r,x_{r+1})\leq S$ and therefore by the above property of δ we have that $\delta(x_0,x_n)\leq \frac{1}{2}S$. □

To finish with the proof of the Theorem we need to show that the metric d is compatible with the topology on G. This follows easily from the fact that (V_n) is a basis at 1, the definition of δ and the inequalities $\delta\geq d\geq \delta/2$. □

The argument provided in the proof of 10 is very useful. Variations and refinements of this argument can often be used in order to characterize a fixed class of Polish groups by whether all its elements admit a left invariant metric which satisfies certain properties. We will see examples in Theorem **??**, Theorem **??**. Another example is the subject of Project 2:

Project 2. Simplify/conceptualize the proof of the theorem: let G be a Polish group. The family of continuous representation of G on reflexive Banach spaces generates the topology on G if and only if G admits a left invariant metric d that is stable, i.e., for all d-bounded sequences $(g_n), (h_k)$ we have that

$$\lim_n \lim_k d(g_n, h_k) = \lim_k \lim_n d(g_n, h_k),$$

whenever the two limits exist.

Lemma 11. *Let G be a Polish group and let d, ρ both be compatible left invariant metrics on G. Then, a sequence (g_n) is d-Cauchy if and only if it is ρ-Cauchy.*

Proof. Assume that (g_n) is d-Cauchy and let $\varepsilon > 0$. We will find n_0 so that for all $m, l \geq n_0$ we have that $\rho(g_m, g_l) < \varepsilon$. Since both d, ρ are compatible with the topology, we can find a $\delta > 0$ so that $B_d(1, \delta) \subseteq B_\rho(1, \varepsilon)$. Let now n_0 so that for all $m, l \geq n_0$ we have that $d(g_m, g_l) < \delta$ and notice that by left invariance we have

$$d(g_m, g_l) < \delta \implies d(g_l^{-1} g_m, 1) < \delta \implies \rho(g_l^{-1} g_m, 1) < \varepsilon \implies \rho(g_m, g_l) < \varepsilon.$$

□

The following definition is essentially a corollary of the previous lemma. The proof that $\widehat{G}^L$ is a a Polish monoid is a sub-argument of the proof of Theorem 8 and will be left to the reader.

Definition 12. Let G be a Polish group. The **left-completion** $\widehat{G}^L$ of G is the completion of G with respect to any compatible left invariant metric. Then multiplication on G extends uniquely on $\widehat{G}^L$ turning $\widehat{G}^L$ into a Polish monoid. The group G is said to be **CLI** if it admits a complete left-invariant metric, i.e. if $\widehat{G}^L = G$.

One can easily compute now $\widehat{G}^L$ when $G = S_\infty$. In particular, consider the space $\mathbb{N}^\mathbb{N}$ of all functions from $\mathbb{N}$ to $\mathbb{N}$ endowed with the pointwise convergence topology. The metric usual metric defined by $d(f, f') = 2^{-n}$ iff $f \neq f'$ and n is the smallest natural number so that $f(n) \neq f'(n)$ is compatible with the topology on $\mathbb{N}^\mathbb{N}$ and it is easy to check that it is complete. This metric restricts to a left invariant metric on $S_\infty \subseteq \mathbb{N}^\mathbb{N}$. Therefore, in order to identify $\widehat{S_\infty}^L$ it suffice to compute the closure of S_∞ inside $\mathbb{N}^\mathbb{N}$ this easily checks out to be the monoid of all injections from $\mathbb{N}$ to $\mathbb{N}$ (which are not surjective in general).

The left completion of the unitary group $\mathcal{U}(\mathcal{H})$ consists of all linear embeddings from $\mathcal{H}$ to $\mathcal{H}$ which preserve the inner product. Similarly, the right completion $\widehat{\mathrm{Homeo}_+([0,1])}^R$ is the collection of all continuous and (non-strictly) increasing surjective maps from $[0, 1]$ to $[0, 1]$.

Theorem 10 implies that every Polish group admits a left invariant metric d_l. As a consequence the metric d_r defined by $d_r(g,h) = d_l(g^{-1}, h^{-1})$ is a compatible right invariant metric. The next example shows that not every Polish group, in fact not every locally compact Polish group, admits a two sided invariant metric. A metric d on G is two sided invariant or simply **invariant** if it is both left and right invariant. Notice that if G admits a compatible invariant metric and $(g_n), (h_n)$ are sequences in G with $\lim g_n h_n = 1$ then $\lim h_n g_n$ is also 1, since

$$d(h_n g_n, 1) = d(h_n^{-1} h_n g_n h_n, h_n^{-1} h_n) = d(g_n h_n) \to 1$$

Example 13. Consider the group topological group $\mathrm{SL}(2,\mathbb{R})$ of all real $2x2$ matrices with determinant 1. The topology it inherits from $\mathbb{R}^4$ makes it a locally compact Polish group. Notice that if

$$A_n = \begin{bmatrix} 1/n & 1/n \\ 0 & n \end{bmatrix}, \quad B_n = \begin{bmatrix} n & 1/n \\ 0 & 1/n \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

then $A_n B_n \to 1$, while $B_n A_n \to C$. Hence by the discussion above $\mathrm{SL}(2,\mathbb{R})$ does not admit an invariant metric.

The following example characterizes all topological groups which admit a compatible invariant metric.

Theorem 14 (Klee)**.** *A topological group G admits an invariant metric compatible with its topology if and only if is Hausdorff and 1 admits a countable neighborhood basis (U_n) at 1 consisting of conjugation invariant sets, i.e., $gU_ng^{-1} = U_n$, for all $g \in G$.*

Proof. Notice that if d is an invariant metric then

$$d(g,1) = d(hg,h) = d(hgh^{-1}, hh^{-1}) = d(hgh^{-1}, 1).$$

Hence, if G admits an invariant metric d then $U_n = B_d(1, 1/n)$ is conjugation invariant.

Conversely, notice that if U is conjugation invariant set then $U^{-1} = \{g^{-1} \mid g \in G\}$ is also conjugation invariant since $\left(hU^{-1}h^{-1}\right)^{-1} = hUh^{-1} = U$, for all $h \in G$. Therefore one can arrange in the proof of Theorem 10 so that the sequence (V_n) is additionally conjugation invariant. As a consequence, in the definition of δ there, if $y^{-1}x \in V_n$ then $hy^{-1}h^{-1}hxh^{-1} \in V_n$. Hence $\delta(x,y) = \delta(hxh^{-1}, hyh^{-1})$ and the same property is inherited by d. We are left to show that d is right invariant, but:

$$d(xh, yh) = d(hxhh^{-1}, hyhh^{-1}) = d(hx, hy).$$

□

In contrast with Theorem 14 notice that in the case of $\mathrm{SL}(2,\mathbb{R})$, if $A_\varepsilon = \begin{bmatrix} 1 & \varepsilon \\ 0 & 1 \end{bmatrix}$ is any fixed element arbitrary close to the identity then we can send it arbitrary far from 1 simply by conjugating it with $B_r = \begin{bmatrix} r & 0 \\ 0 & 1/r \end{bmatrix}$, for large enough r.

Definition 15. A Polish group G is **TSI** is it admits a (two-sided) invariant metric.

We have now the following picture among the various classes of groups we have seen so far:

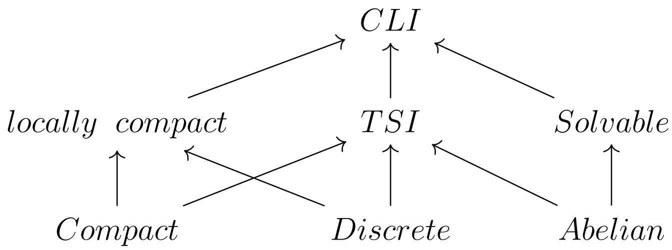


FIGURE 1. The arrows denote strict inclusions in the class of Polish groups

Proposition 16. *All the inclusions in the diagram hold. In particular:*

(1) all compact Polish groups are TSI;
(2) all discrete Polish groups are TSI;
(3) all Abelian Polish groups are TSI;
(4) all locally compact Polish groups are CLI;
(5) all TSI Polish groups are CLI;
(6) all solvable Polish groups are CLI (Hjorth-Solecki);

Proof. (1) By Theorem 10 fix a left invariant metric d on G and define $d_*\colon G\times G\to\mathbb{R}$ by setting

$$d_*(x,y)=\sup_{h\in G}d_*(xh,yh).$$

By compactness, for every x and y the supremum is realized by some actual $h\in G$. It is immediate that d_* is two sided invariant. So we are left to check that it is compatible with the topology. Since (G,τ_d) is compact, it suffice to show that the identity map from (G,τ_d) to (G,τ_{d_*}) is continuous. But notice that if $U=B^{d_*}(x,\varepsilon)$ is the ε-ball around x in (G,τ_{d_*}) then the complement U^c of U is a projection of a compact subset of $(G,\tau_d)\times(G,\tau_d)$ to (G,τ_d):

$$U^c=\mathrm{proj}_G\{(y,h)\in G\times G\mid d(xh,yh)\geq\varepsilon\}.$$

Therefore U is open in (G,τ_d) as well.

(2) take $d(x,y)=1\iff x\neq y$.

(3) Immediate by Theorem 14.

(4) By Theorem 10 fix a left invariant metric d on G. Notice that if (g_n) is a d-Cauchy sequence then there is some compact set $K\subseteq G$ so that for all (g_n) eventually lies in K and therefore it converges to some $g\in K$.

(5) Let d is an invariant metric on G. By corollary 9 we have that D is a complete metric on G. But D is left-invariant in this case since by invariance of d we have that $d(g^{-1},h^{-1})=d(g,h)$

(6) If G is solvable then it built by a finite sequence of abelian extensions starting from the trivial group 1. Hence this follows by induction using (3),(5) and the next theorem. □

Theorem 17 (Gao)**.** *Let G be a Polish group and let H be a closed normal subgroup of G. The following are equivalent:*

(1) G is CLI;
(2) Both H and G/H are CLI.

We will not prove this theorem but it will be important to recall some facts about quotients of Polish groups. Let G be a Polish group and let H be any subgroup of it. We denote by G/H the collection $\{Hg \mid g \in G\}$ of all right cosets of H. If $\pi \colon G \to G/H$ is the natural projection $g \mapsto Hg$ then a set $U \subseteq G/H$ is open in the quotient topology if $\pi^{-1}(U)$ is open in G. This makes π both continuous and open. The topology is bad in general when H is not closed. However, when H is closed G/H is Polish. This follows from the following classical theorem:

Theorem 18 (Sierpinski)**.** *If X is a Polish space, Y a metrizable space and $\pi \colon X \to Y$ a continuous and open surjection. Then Y is Polish.*

It is not difficult to see that G/H is metrizable when H is closed but one can construct explicitly a metric d^* on G/H as follows:

Proposition 19. *If d is a left invariant metric on the Polish group G and H is a closed subgroup of H then d^* is a compatible metric on G/H where*

$$d(Hx, Hy) = \inf\{d(g_x, g_y) \mid g_x \in Hx,\ g_y \in Hy\}.$$

It is a good exercise working out the details of the proof of the above proposition and trace the reasons as to why we use the **left** invariant metric d to define d^* on space of **right** cosets.

3. Week 3

Recall that we view the Polish group S_∞ as a G_δ subset of $\mathbb{N}^\mathbb{N}$ endowed with the pointwise convergence topology. This topology is compatible with the metric given by $d(g,h) = 2^{-n}$ iff $g \neq h$ and n is the smallest natural number so that $g(n) \neq h(n)$, which is a left invariant metric on S_∞. This metric is an ultrametric, i.e., it satisfies the following strengthening of the triangle inequality.

A metric d is an **ultrametric** if and only if for all x, y, z we have that

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}.$$

We say that a Polish group is **non-Archimedean** if it admits a compatible left invariant ultrametric. There are many equivalent reformulations of this property.

Theorem 20. *Let G be a Polish group. The following are equivalent:*

(1) G is isomorphic to a closed subgroup of S_∞;
(2) G admits a compatible left invariant ultrametric;
(3) G admits a countable neighborhood basis at 1 consisting of open subgroups;
(4) G admits a countable basis for the topology $\mathcal{B}$ that is invariant under multiplication from the left, i.e., $gB \in \mathcal{B}$ for all $B \in \mathcal{B}$ and $g \in G$.

Proof. (1) $\to$ (2) is immediate since we can always restrict the aforementioned left invariant ultrametric of S_∞ on G.

For (2) $\to$ (3), notice that if $d(g,1), d(h,1) < \varepsilon$ then

$$d(gh,1) = d(h, g^{-1}) \leq \max\{d(h,1), d(g^{-1},1)\} = \max\{d(h,1), d(1,g)\} < \varepsilon.$$

Therefore, any countable collection of vanishing open d-balls around the identity will do.

For (3) $\to$ (4) let $\mathcal{B}_1$ be a countable neighborhood basis at 1 consisting of open subgroups and notice that each $B \in \mathcal{B}_1$ has countably many left cosets. Hence the required basis is the collection $\mathcal{B} = \{gB \mid g \in G,\ B \in \mathcal{B}_1\}$.

For (4) $\to$ (1) we enumerate $\mathcal{B} = \{B_0, B_1, \ldots\}$ and we consider the representation $\pi\colon G \to S_\infty$ given by

$$\pi(g)(k) = l \iff gB_k = B_l.$$

It is immediate that π is a homomorphism. It is injective since $\mathcal{B}$ separates points and continuous since $gB_k = B_l$ is witnessed by a whole open set, namely $g^{-1}B_k$. So we are left to show that π is an open map from G to $\pi(G)$. This will show that G is isomorphic to $\pi(G)$ and that $\pi(G)$ is Polish. By Lemma 3 the latter it implies that $\pi(G)$ is closed.

Let $g \in B_l \in \mathcal{B}$. It suffice to find an open set $V \subseteq S_\infty$ with $\pi(g) \in V \cap \pi(G) \subseteq \pi(B_l)$. Let $B_k = g^{-1}B_l \in \mathcal{B}$ and set $V = \{f \in S_\infty \mid f(k) = l\}$. Then V is clearly open with $\pi(g) \in V$. Moreover, if $h \in G$ with $\pi(h) \in V$ then $hB_k = B_l \implies h \in B_l$. Hence $V \cap \pi(G) \subseteq \pi(B_l)$. □

Corollary 21. *If (G_n) is a countable collection of closed subgroups of S_∞ then the group $\Pi_n G_n$ is isomorphic to a closed subgroup of S_∞. In particular, $\mathbb{Z}^\omega$ is a closed subgroup of S_∞.*

Theorem 20 contained many equivalent "topological" reformulations of non-Archimedeanity. The next result allows us to associate to each non-Archimedean group G a countable structure M so that $G = \mathrm{Aut}(M)$. First recall some definitions.

A **language** $\mathcal{L} = \{R, S, \ldots, f, g, \ldots\}$ is a collections of relational symbols $R, S, \ldots$ and function symbols $f, g, \ldots$, together with the arity function $\alpha\colon \mathcal{L} \to \mathbb{N}$ which assigns to each symbol $s \in \mathcal{L}$ the size of the tuple $\bar{x}$ with respect to which $s(\bar{x})$ is going to be treat as syntactically correct.

An $\mathcal{L}$-structure $\boldsymbol{M} = (M; R^M, S^M, \ldots; f^M, g^M, \ldots)$ consists of a set M together with interpretations for every every symbol in $\mathcal{L}$. In particular if R is a relation in $\mathcal{L}$ with arity $\alpha(R) = n$ then the **interpretation** R^M of R in $\boldsymbol{M}$ is just a subset of M^n. Similarly the **interpretation** f^M of an n-ary function f in $\boldsymbol{M}$ is just a function from M^n to M.

The function symbols $c \in \mathcal{L}$ whose arity is 0 we call them constants and c^M is by definition some point in $\boldsymbol{M}$. Since the 0-ary Cartesian product M^0 of any set M is the set $\{\emptyset\}$ there are only 2 possible interpretations of a 0-ary relation symbol which we usually denote by $\top$ and $\bot$. Often when the context is set we will simplify our notation and use the letters $R, f, \ldots$ instead of $R^M, f^M, \ldots$

Examples.

- We can view the set of rationals endowed with the usual linear order as an $\mathcal{L}$-structure $(\mathbb{Q}, \leq)$ where $\mathcal{L}$ consists of one relation symbol $\leq$ with arity $\alpha(\leq) = 2$.
- Any countable discrete group Γ can be viewed as a a countable $\mathcal{L}$-structure $(\Gamma, 1, ()^{-1}, *)$ where $\mathcal{L}$ consists of three function symbols with $\alpha 1 = 0$, $\alpha(()^{-1}) = 1$, and $\alpha(*) = 2$.

Given an $\mathcal{L}$-structure $\boldsymbol{M}$ we denote by $\mathrm{Aut}(\boldsymbol{M})$ the group of all **automorphisms** of $\boldsymbol{M}$ under composition. Recall that that $\varphi \in \mathrm{Aut}(\boldsymbol{M})$ is f is a bijection $f\colon M \to M$ so that for every relation symbol $R \in \mathcal{L}$ and any tuple $\bar{a} = (a_1, \ldots, a_n)$ in M we have that

$$(a_1, \ldots, a_n) \in R^M \iff (\varphi(a_1), \ldots, \varphi(a_n)) \in R^M,$$

and for every function symbol $f \in \mathcal{L}$, any tuple $\bar{a} = (a_1, \ldots, a_n)$ in M and $b \in M$ we have that

$$f^M(a_1, \ldots, a_n) = b \iff f^M(\varphi(a_1), \ldots, \varphi(a_n)) = \varphi(b)$$

We view $\mathrm{Aut}(\boldsymbol{M})$ as a topological group with the pointwise convergence topology. One of the implications of the next theorem is that if $\boldsymbol{M}$ is a countable structure then $\mathrm{Aut}(\boldsymbol{M})$ is Polish.

Theorem 22. *Let G be a Polish group. The following are equivalent:*

(1) G is non-Archimedean;

(2) *There is a countable structure* $\boldsymbol{M}$ *on a relational language and* $G \simeq \mathrm{Aut}(\boldsymbol{M})$.

Proof. If $\boldsymbol{M}$ is a countable $\mathcal{L}$-structure then by enumerating $\boldsymbol{M}$ we can assume that $\mathrm{Aut}(\boldsymbol{M})$ is a subgroup of S_∞. To see that it is closed notice that if $f \in S_\infty \setminus \mathrm{Aut}(\boldsymbol{M})$ then there is some relation symbol $R \in \mathcal{L}$ (or function symbol, but this is left as an exercise) and a tuple $\bar{a}$ in M so that $\bar{a} \in R^M$ but not $\varphi(\bar{a}) \in R^M$ (or, not $\bar{a} \in R^M$ but $\varphi(\bar{a}) \in R^M$, but this is left to the reader). But then any $g \in S_\infty$ in the open subset defined by the data $\bar{a} \mapsto \varphi(\bar{a})$ will fail to be an automorphism.

Assume now that G is non-Archimedean, i.e. by Theorem 20, a closed subgroup of S_∞. For every $n \geq 1$ let $\{O^n_k, k \in I\}$ with $O^n_k \subseteq \mathcal{N}^n$, be the collection of all orbits in the diagonal action of G on $\mathbb{N}^n$. For each such orbit introduce an n-ary relation R^n_k and define a structure $\boldsymbol{M}$ by setting the interpretation $(R^n_k)^M$ of R^n_k in $\boldsymbol{M}$ to be O^n_k. It is immediate that $G \subseteq \mathbb{A}\approxeq\approx(\boldsymbol{M})$ for the converse inclusion use the fact that G is closed. □

The structures $\boldsymbol{M}$ provided by Theorem 22 enjoy a very useful property know as *ultrahomogeneity*. In short, any isomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ between finite substructures of $\boldsymbol{M}$ extends to a global automorphism $\tilde{f} \in \mathrm{Aut}(\boldsymbol{M})$. Many natural examples fail to be ultrahomogeneous. For example the substructures $(\{0,1\}, \leq)$ and $(\{0,2\}, \leq)$ of $(\mathbb{Z}, \leq)$ are isomorphic but this isomorphism cannot be extended to an element of $\mathrm{Aut}(\mathbb{Z}, \leq)$. To put things into context we need some definitions.

Rather than studying the various mathematical structures directly, i.e., up to isomorphism, logic is often interested in comparing them by means of the formal statements they satisfy. The usual first order logic cares about finitary statements, i.e., $\mathcal{L}$-formulas. However when one wants to study countable structures up to isomorphism it sometimes is convenient to rely on infinitary statemetns known as $\mathcal{L}_{\omega_1,\omega}$-formulas. Both logics are instances of the following definition where for historic and convenience purposes one writes $\mathcal{L}$ for $\mathcal{L}_{\aleph_0,\omega}$ and $\mathcal{L}_{\omega_1,\omega}$ for instead of $\mathcal{L}_{\aleph_1,\omega}$.

Definition 23. Let $\mathcal{L}$ be a language and let κ be an infinite cardinal. Let also X be a collection of variables.The set $\Phi_{\kappa,\omega}(\mathcal{L})$ of $\mathcal{L}_{\kappa,\omega}$**-formulas** is the smallest collection of statements so that:

(1) $R(x_1, \ldots, x_n)$ and $f(x_1, \ldots, x_n = x_{n+1})$ are in $\Phi_{\kappa,\omega}(\mathcal{L})$, for all $R, f \in \mathcal{L}$;
(2) if $\varphi(\bar{x})$ is in $\Phi_{\kappa,\omega}(\mathcal{L})$ then $\neg\varphi(\bar{x})$ is in $\Phi_{\kappa,\omega}(\mathcal{L})$;
(3) if if $\varphi(\bar{x}, y)$ is in $\Phi_{\kappa,\omega}(\mathcal{L})$ then $\exists y \varphi(\bar{x}, y)$ and $\forall y \varphi(\bar{x}, y)$ are in $\Phi_{\kappa,\omega}(\mathcal{L})$;
(4) If Ψ is a collection of $\mathcal{L}_{\kappa,\omega}$-formulas with $|\Psi| < \kappa$ and so that there is a finite tuple $\bar{x} = (x_1, \ldots, x_n)$ of free variables so that the free variables of each $\phi \in \Psi$ are among $\bar{x}$, then the following are also $\mathcal{L}_{\kappa,\omega}$-formulas:

$$\bigwedge_{\phi \in \Psi} \phi(\bar{x}), \quad \bigvee_{\phi \in \Psi} \phi(\bar{x}).$$

A **quantifier free $\mathcal{L}_{\kappa,\omega}$-formula** is any $\mathcal{L}_{\kappa,\omega}$-formula that is constructed without any use of the the operation (3) above. A **$\mathcal{L}_{\kappa,\omega}$-sentence** is any $\mathcal{L}_{\kappa,\omega}$-formula σ containing no free variables.

If $\varphi(\bar{x})$ is some $\mathcal{L}_{\kappa,\omega}$-formulas and $\bar{a}$ is a tuple in an $\mathcal{L}$-structure $\boldsymbol{M}$ the we will write $\boldsymbol{M} \models \varphi(\bar{a})$ to denote that $\bar{a}$ satisfies the statement $\varphi(\bar{x})$ in $\boldsymbol{M}$. This can be defined formally by induction on the complexity of the formula in the obvious way. The **$\mathcal{L}_{\kappa,\omega}$-theory** of $\boldsymbol{M}$ is the collection $\mathrm{Th}_{\kappa,\omega}(\boldsymbol{M})$ of all $\mathcal{L}_{\kappa,\omega}$-sentences satisfied by $\boldsymbol{M}$.

Let $\bar{a} = (a_1, \ldots, a_n)$ be a tuple in the $\mathcal{L}$-structure $\boldsymbol{M}$ and let $\bar{b} = (b_1, \ldots, b_n)$ be a tuples in $\boldsymbol{N}$. We write $(\boldsymbol{M}, \bar{a}) \equiv_{qf} (\boldsymbol{N}, \bar{b})$ if the tuples $\bar{a}$ and $\bar{b}$ have the same **quantifier free type**, i.e., if every quantifier free formula is satisfied by $\bar{a}$ if and only if it is satisfied by $\bar{b}$. Notice that we haven't specified a κ since this doesn't depend on which $\mathcal{L}_{\kappa,\omega}$ logic we chose. In fact $(\boldsymbol{M}, \bar{a}) \equiv_{qf} (\boldsymbol{N}, \bar{a})$ holds if and only if the substructure $\langle \bar{a} \rangle_{\boldsymbol{M}}$ generated by $\bar{a}$ in $\boldsymbol{M}$ is isomorphic to the substructure $\langle \bar{b} \rangle_{\boldsymbol{N}}$ generated by $\bar{b}$ in $\boldsymbol{N}$, by an isomorphism that sends a_i to b_i. Similarly we write $(\boldsymbol{M}, \bar{a}) \equiv_{\mathcal{L}} (\boldsymbol{N}, \bar{b})$ if $(\boldsymbol{M}, \bar{a})$ and $(\boldsymbol{N}, \bar{b})$ are **elementarily** equivalent, i.e., if $\bar{a}$ and $\bar{b}$ satisfy the the same $\mathcal{L}$-formulas ($\mathcal{L}_{aleph0,\omega}$-formulas in the definition). Finally we write $(\boldsymbol{M}, \bar{a}) \equiv_{\omega_1,\omega} (\boldsymbol{N}, \bar{b})$ if $(\boldsymbol{M}, \bar{a})$ and $(\boldsymbol{N}, \bar{b})$ are **$\mathcal{L}_{\omega_1,\omega}$-elementarily** equivalent, i.e., if $\bar{a}$ and $\bar{b}$ satisfy the the same $\mathcal{L}_{\omega_1,\omega}$-formulas ($\mathcal{L}_{\aleph_1,\omega}$-formulas in the definition).

Definition 24. Let $\boldsymbol{M}$ be a countable $\mathcal{L}$-structure. We say that $\boldsymbol{M}$ is **ultrahomogeneous** if for every two finite tuples $\bar{a}, \bar{b}$ in $\boldsymbol{M}$, is $(\boldsymbol{M}, \bar{a}) \equiv_{qf} (\boldsymbol{M}, \bar{b})$ then there is a $f \in \mathrm{Aut}(\boldsymbol{M})$ with $f\bar{a} = \bar{b}$. We say that $\boldsymbol{M}$ is **homogeneous** if for every two finite tuples $\bar{a}, \bar{b}$ in $\boldsymbol{M}$, is $(\boldsymbol{M}, \bar{a}) \equiv_{\mathcal{L}} (\boldsymbol{M}, \bar{b})$ then there is a $f \in \mathrm{Aut}(\boldsymbol{M})$ with $f\bar{a} = \bar{b}$.

One could define in an analogous homogeneity property for tuples which satify the stronger assumption that $(\boldsymbol{M}, \bar{a}) \equiv_{\omega_1,\omega} (\boldsymbol{M}, \bar{b})$. However in Theorem 25 we will see that the assumption $(\boldsymbol{M}, \bar{a}) \equiv_{\omega_1,\omega} (\boldsymbol{M}, \bar{b})$ is too strong.

Examples.

(1) The structure $(\mathbb{Q}, \leq)$ is ultrahomogeneous. One can see this by means of a standard *back and forth* argument: let $\bar{a}, \bar{b}$ be two tuples in $\mathbb{Q}$ which satisfy the same quantifier free type and fix some enumeration of $\mathbb{Q}$. Step by step we will extend the assignment $\bar{a} \mapsto \bar{b}$ to a bijection from $\mathbb{Q}$ to $\mathbb{Q}$ which preserves $\leq$. Let $\bar{a}, c_1, \ldots, c_{n-1} \mapsto \bar{b}, d_1, \ldots, d_{n-1}$ be the assignment before the step n. If n is odd, let c_n be the first element (with respect to the fixed enumeration) in $\mathbb{Q} \setminus \{\bar{a}, c_1, \ldots, c_{n-1}\}$. Since $(\mathbb{Q}, \leq)$ is dense and without endpoints we can easily find d_n so that $(\bar{a}, c_1, \ldots, c_{n-1}, c_n) \equiv_{qf} (\bar{b}, d_1, \ldots, d_{n-1}, d_n)$. If n is even, let d_n be the first element $\mathbb{Q} \setminus \{\bar{b}, d_1, \ldots, d_{n-1}\}$ and again extend by finding appropriate c_n. Odd steps guarantee that the resulting map will be total and even steps guarantee that it will be surjective.

(2) The structure $(\mathbb{Z}, \leq)$ is not ultrahomogeneous however it is homogeneous. One can easily see this since the distance $|n - k|$ of two integers is first

order definable in $(\mathbb{Z},\leq)$ and automorphisms of $(\mathbb{Z},\leq)$ are precisely the order preserving bijections which additionally preserve distance.

(3) The structure $\boldsymbol{M}=(\mathbb{Z},\leq)\oplus(\mathbb{Z},\leq)$ is not homogeneous, where $(\mathbb{Z},\leq)\oplus(\mathbb{Z},\leq)$ is has domain the set $\mathbb{Z}\times\{0\}\bigcup\mathbb{Z}\times\{1\}$, for each $i=0,1$ the elements within $\mathbb{Z}\times\{i\}$ are ordered as in $(\mathbb{Z},\leq)$ and each element in $\mathbb{Z}\times\{0\}$ is smaller that every element of $\mathbb{Z}\times\{1\}$. In order to see this one has to use the "quantifier elimination" machinery in order to prove that $(\boldsymbol{M},(0,0))\equiv_{\mathcal{L}}(\boldsymbol{M},(0,1))$. Given this it is easy to see why there is no automorphism sending the 0 of the first copy to the 0 of the second copy.

Theorem 25 (Karp)**.** *Let $\boldsymbol{M}$ be a countable $\mathcal{L}$-structure and let $\bar{a},\bar{b}$ be tuples in $\boldsymbol{M}$. Then, $(\boldsymbol{M},\bar{a})\equiv_{\omega_1\omega}(\boldsymbol{M},\bar{b})$ if and only if there is an automorphism f of $\boldsymbol{M}$ sending $\bar{a}$ to $\bar{b}$.*

Proof. If f is an automorphism of $\boldsymbol{M}$ with $f\bar{a}=\bar{b}$ then it immediate to see that $(\boldsymbol{M},\bar{a})\equiv_{\omega_1\omega}(\boldsymbol{M},\bar{b})$. Formally one can use straight forward induction on the complexity on each $\mathcal{L}_{\omega_1\omega}$-formula.

Conversely, notice that if $(\boldsymbol{M},\bar{a})\equiv_{\omega_1\omega}(\boldsymbol{M},\bar{b})$ then for every c in M there is a d in M so that $(\boldsymbol{M},\bar{a},c)\equiv_{\omega_1\omega}(\boldsymbol{M},\bar{b},d)$ because if not then for every $d\in M$ we can find some $\mathcal{L}_{\omega_1\omega}$-formula $\phi_d(\bar{x},y)$ so that

$$\boldsymbol{M}\models\phi_d(\bar{a},c)\quad\text{but}\quad\boldsymbol{M}\models\neg\phi_d(\bar{b},d),$$

and this would contradict $(\boldsymbol{M},\bar{a})\equiv_{\omega_1\omega}(\boldsymbol{M},\bar{b})$ since then we have

$$\boldsymbol{M}\models\exists y\bigwedge_{d\in M}\phi_d(\bar{a},y)\quad\text{but}\quad\boldsymbol{M}\not\models\exists y\bigwedge_{d\in M}\phi_d(\bar{b},y),$$

Hence inductively we can produce a back and forth system resulting to an automorphism of $\boldsymbol{M}$ sending $\bar{a}$ to $\bar{b}$. □

Exercise. Check that the structures we constructed in the proof of Theorem 22 are actually ultrahomogeneous. Given any $\mathcal{L}$-structure $\boldsymbol{M}$, explain how one can define a language $\mathcal{L}'$ with $\mathcal{L}\subseteq\mathcal{L}'$ together with appropriate interpretations for the new symbols that will extend $\boldsymbol{M}$ to an ultrahomogenous $\mathcal{L}'$ structure $\boldsymbol{M}'$ (on the same domain with $\boldsymbol{M}$).

We conclude with the following characterization of CLI non-Archimedean groups due to Su Gao.

Theorem 26 (Gao)**.** *Let $\boldsymbol{M}$ be a countable structure. The following are equivalent:*

(1) $\mathrm{Aut}(\boldsymbol{M})$ *is CLI;*
(2) there are no uncountable structures satisfying the $\mathcal{L}_{\omega_1\omega}$-theory of $\boldsymbol{M}$.

Before we proceed to the proof notice the immense difference of $\mathcal{L}_{\omega_1\omega}$-logic with the first order logic where compactness theorem guarantees that if $\boldsymbol{M}$ is infinite then the $\mathcal{L}$-theory of $\boldsymbol{M}$ admits models in every cardinal. For the proof we first need a lemma.

Lemma 27. *Let $\boldsymbol{M}$ be a countable structure. A map $\gamma\colon M \to M$ is in left completion $\widehat{\mathrm{Aut}(\boldsymbol{M})}^{L}$ if and only if γ is an $\mathcal{L}_{\omega_1\omega}$- elementary embedding from $\boldsymbol{M}$ to $\boldsymbol{M}$, i.e., γ is an embedding and $(\boldsymbol{M},\bar{a}) \equiv_{\omega_1\omega} (\boldsymbol{M},\gamma\bar{a})$ for every finite $\bar{a}$ in M.*

Proof. Assume that there exists left-Cauchy sequence (g_n) in $\mathrm{Aut}(\boldsymbol{M})$ converging to γ and let $\bar{a}$ be any finite tuple in M. Then we can find n large enough so that $g_n\bar{a} = \gamma\bar{a}$. Hence

$$(\boldsymbol{M},\gamma\bar{a}) \equiv_{\omega_1\omega} (\boldsymbol{M},g_n\bar{a}) \equiv_{\omega_1\omega} (\boldsymbol{M},\bar{a}),$$

where the last equivalence holds because g_n is an automorphism.

Conversely, let γ be an $\mathcal{L}_{\omega_1\omega}$- elementary embedding from $\boldsymbol{M}$ to $\boldsymbol{M}$. To build a left-Cauchy sequence converging to γ it suffice to find for every finite tuple $\bar{a}$ in M an automorphism g so that $g\bar{a} = \gamma\bar{a}$. But since $(\boldsymbol{M},\bar{a}) \equiv_{\omega_1\omega} (\boldsymbol{M},\gamma\bar{a})$ the same argument as in the proof of Theorem 25 produces a back and forth system from M to M starting from the assignment $\bar{a} \mapsto \gamma\bar{a}$. The result of this back and forth system is the required automorphism. □

Sketch of proof of Theorem 26. We briefly sketch the proof relying on some facts regarding $\mathcal{L}_{\omega_1\omega}$-logic which we will use without proving them.

The first fact is Scott's isomorphism theorem:

"If $\boldsymbol{M}$ is a countable $\mathcal{L}$ structure then there is a single $\mathcal{L}_{\omega_1\omega}$-sentence σ so that for every other countable $\mathcal{L}$ structure $\boldsymbol{N}$ we have that $\boldsymbol{N} \models \sigma$ if and only if $\boldsymbol{M}$ and $\boldsymbol{N}$ are isomorphic".

The second fact is the following analogue of what is model theory is known downward Löwenhein-Skolem theorem:

"If $\mathcal{L}$ is a countable language. If $\boldsymbol{N}$ is any infinite structure and let T be a countable $\mathcal{L}_{\omega_1\omega}$-theory, then for any countable subset A of N there exists a countable substructure $\boldsymbol{M}$ of $\boldsymbol{N}$ with $A \subseteq M$ so that the inclusion $i\colon \boldsymbol{M} \to \boldsymbol{N}$ is elementary with respect to $\mathcal{L}_{\omega_1\omega}$-subformulas of sentences of T, i.e., if $\varphi(\bar{x})$ is a subformula appearing within some $\sigma \in T$ then for all $\bar{a}$ in M we have that $\boldsymbol{M} \models \varphi(\bar{a})$ if and only if $\boldsymbol{N} \models \varphi(\bar{a})$."

See [Kue, Theorem 2.3 and Theorem 1.1] for proofs—in the more general $\mathcal{L}_{\kappa\lambda}$ setting—of these statements.

Let σ be the Scott sentence of $\boldsymbol{M}$ and assume that $\boldsymbol{N}$ is an uncountable structure satisfying σ. By downward Löwenhein-Skolem theorem applied on $\boldsymbol{N}$ for $T = \{\sigma\}$ and $A = \emptyset$ we get a substructure $\boldsymbol{M}_0$ of $\boldsymbol{N}$ that is countable so that so that the inclusion $i_0\colon \boldsymbol{M}_0 \to \boldsymbol{N}$ is elementary with respect to $\mathcal{L}_{\omega_1\omega}$-subformulas of sentences of T. Since T is a complete $\mathcal{L}_{\omega_1\omega}$-theory it follows that i_0 is an elementary $\mathcal{L}_{\omega_1\omega}$-embedding. By a second application of downward Löwenhein-Skolem theorem applied on $\boldsymbol{N}$ for $T = \{\sigma\}$ and A strictly including M_0 we get a countable substructure $\boldsymbol{M}_1$ of $\boldsymbol{N}$ so that, as above, the inclusion $i_1\colon \boldsymbol{M}_1 \to \boldsymbol{N}$ is elementary $\mathcal{L}_{\omega_1\omega}$-embedding. But then both $\boldsymbol{M}_0, \boldsymbol{M}_1$ satisfy the same Scott sentence and therefore they are isomorphic to $\boldsymbol{M}$. Moreover the map $\gamma\colon \boldsymbol{M}_0 \to \boldsymbol{M}_1$ with $\gamma = i_0$ is an elementary $\mathcal{L}_{\omega_1\omega}$-embedding since

$$(\boldsymbol{M}_0, \gamma\bar{a}) \equiv_{\omega_1\omega} (\boldsymbol{N}, i_0\bar{a}) \equiv_{\omega_1\omega} (\boldsymbol{M}_1, \gamma\bar{a}),$$

for every $\bar{a}$ in M_0. By Lemma 27 $\gamma \in \widehat{\mathrm{Aut}(\boldsymbol{M})}^L$, and since M_0 is strictly contained in M_1 we have that $\gamma \notin \mathrm{Aut}(\boldsymbol{M})$.

Conversely, if $\gamma \in \widehat{\mathrm{Aut}(\boldsymbol{M})}^L \setminus \mathrm{Aut}(\boldsymbol{M})$ then by Lemma 27 γ is an elementary $\mathcal{L}_{\omega_1\omega}$-embedding that is not surjective. By transfinite induction we build an ω_1-sequence of elementary $\mathcal{L}_{\omega_1\omega}$-embeddings $\gamma^{\kappa}_{\lambda}\boldsymbol{M}_\kappa \to \boldsymbol{M}_\lambda$, $\kappa \leq \lambda < \omega_1$ so that $\boldsymbol{M}_\kappa$ is isomorphic to $\boldsymbol{M}$ and each embedding is strict. Given this it is easy to check that the colimit of all these structures is uncountable and that is satisfies the same $\mathcal{L}_{\omega_1\omega}$-theory as $\boldsymbol{M}$. For the construction, if $\boldsymbol{M}_\kappa$ has been defined and it is isomorphic to $\boldsymbol{M}$, we define $\boldsymbol{M}_{\kappa+1}$ and $\gamma^{\kappa}_{\kappa+1}$ so that $\gamma^{\kappa}_{\kappa+1} \colon \boldsymbol{M}_\kappa \to \boldsymbol{M}_{\kappa+1}$ is isomorphic to $\gamma \colon \boldsymbol{M} \to \boldsymbol{M}$. If λ is a countable limit ordinal we define $\boldsymbol{M}_\lambda$ to be the colimit of what we have so far and $\gamma^{\kappa}_{\lambda} \colon \boldsymbol{M}_\kappa \to \boldsymbol{M}_\lambda$ the induced colimit maps. It is easy to check that each $\gamma^{\kappa}_{\lambda}$ is elementary $\mathcal{L}_{\omega_1\omega}$-embedding since for every $\overline{a}$ in $\boldsymbol{M}_\lambda$ there should be $\kappa < \lambda$ with $\overline{a} \in \boldsymbol{M}_\lambda$. In particular $\boldsymbol{M}_\lambda$ is countable and satisfies the same Scott sentence as each previous $\boldsymbol{M}_\kappa$. By Scott isomorphism theorem it is isomorphic to $\boldsymbol{M}$. □

4. Week 4 (the week of shame)

One of the most effective tools for analyzing closed subgroups of S_∞ is the theory of Fraïssé limits. We will see that *Fraïssé limit* is just another term for countable ultrahomogeneous structure. Although it is not necessary, it is definitely convenient to restrict our attention to purely relational languages $\mathcal{L}$. For purely relational languages there is bijective correspondence between subsets A of an $\mathcal{L}$-structure and substructures $\boldsymbol{A} = \langle A \rangle_{\boldsymbol{M}}$. As a consequence a relational structure $\boldsymbol{M}$ is *ultrahomogeneous* if whenever $g \colon \boldsymbol{A} \to \boldsymbol{B}$ is an isomorphism between **finite** substructures of $\boldsymbol{M}$ then g extends to an automorphism of $\boldsymbol{M}$. Since the proof of Theorem 22 associates to each closed subgroup of S_∞ a countable purely relational ultrahomogenous structure, this restriction will not make us lose in generality.

The big advantage in working with an ultrahomogeneous structure $\boldsymbol{M}$ boils down to the fact that most of the information regarding the symmetries of $\boldsymbol{M}$ is directly encoded on the isomorphism type of its finite substructures (local data) and not on the way that these substructures sit inside $\boldsymbol{M}$ (global data): if $\boldsymbol{A}$ is a finite structure, it doesn't matter how you embed it within $\boldsymbol{M}$ because all these embeddings can be exchanged by an automorphism. A reflection of this high symmetry is that ultrahomogeneous structures can be pieced together via a very canonical limiting process known as "taking the Fraïssé limit."

Definition 28. Let $\boldsymbol{M}$ be an $\mathcal{L}$-structure (in a relational language). The age of $\boldsymbol{M}$, denoted by $\mathrm{Age}(\boldsymbol{M})$, is the collection of all finite $\mathcal{L}$-structures which embed in $\boldsymbol{M}$.

Notice that the age of any structure is a class (not a set) which we view as a category where arrows are embeddings. For example, $\mathrm{Age}((\mathbb{Q}, \leq))$ is the class of all finite linear orderings.

We write $\boldsymbol{A} \leq \boldsymbol{B}$ when $\boldsymbol{A}$ embeds in $\boldsymbol{B}$ and $\boldsymbol{A} \leq_f \boldsymbol{B}$ when we want to keep track of a particular map $f\colon A \to B$ that is witnessing this embedding.

Lemma 29. *Let $\boldsymbol{M}$ be a countable $\mathcal{L}$-structure (in a relational language). Then:*

(1) $\mathrm{Age}(\boldsymbol{M})$ *is countable up to isomorphism;*
(2) $\mathrm{Age}(\boldsymbol{M})$ *satisfies the Hereditary Property (HP), i.e., if* $\boldsymbol{B} \in \mathrm{Age}(\boldsymbol{M})$ *and* $\boldsymbol{A} \leq \boldsymbol{B}$ *then* $\boldsymbol{A} \in \mathrm{Age}(\boldsymbol{M})$*;*
(3) $\mathrm{Age}(\boldsymbol{M})$ *satisfies the Joint Embedding Property (JEP), i.e., if* $\boldsymbol{B}, \boldsymbol{C} \in \mathrm{Age}(\boldsymbol{M})$ *then there is* $\boldsymbol{D} \in \mathrm{Age}(\boldsymbol{M})$ *so that* $\boldsymbol{B}, \boldsymbol{C} \leq \boldsymbol{D}$.

If moreover $\boldsymbol{M}$ is ultrahomogeneous, then additionally

(4) $\mathrm{Age}(\boldsymbol{M})$ *satisfies the Amalgamation Property (AP), i.e., if* $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathrm{Age}(\boldsymbol{M})$ *and* $\boldsymbol{A} \leq_f \boldsymbol{B}$*,* $\boldsymbol{A} \leq_g \boldsymbol{C}$*, then there is* $\boldsymbol{D} \in \mathrm{Age}(\boldsymbol{M})$ *and* $\boldsymbol{B} \leq_{f'} \boldsymbol{D}$*,* $\boldsymbol{C} \leq_{g'} \boldsymbol{D}$ *so that* $f' \circ f = g' \circ g$.

Proof. For (4), let $\boldsymbol{B}, \boldsymbol{C}$ be realized as substructures of $\boldsymbol{M}$ and let $\boldsymbol{A}_B$ and $\boldsymbol{A}_C$ be the copies of $\boldsymbol{A}$ within these two fixed substructures on $\boldsymbol{M}$ (realizing $\boldsymbol{A} \leq_f \boldsymbol{B}$ and $\boldsymbol{A} \leq_g \boldsymbol{C}$). By ultrahomogeneity we can find $\varphi \in \mathrm{Aut}(\boldsymbol{M})$ so that $\varphi(\boldsymbol{A}_B) = \boldsymbol{A}_C$. Let $\boldsymbol{D}$ be the structure generated by $\varphi(\boldsymbol{B}) \cup \boldsymbol{C}$ and f', g' be the obvious embeddings. □

Definition 30. A **Fraïssé class** is any collection $\mathcal{K}$ of finite $\mathcal{L}$-structures satisfying the properties (1)–(4) from the above lemma.

The following theorem is an "inverse" to Lemma 29.

Theorem 31. *Let $\mathcal{K}$ be a Fraïssé class of finite $\mathcal{L}$-structures. Then there exists a unique up to isomorphism countable structure $\boldsymbol{M}$ which satisfies any of the following equivalent properties:*

(1) $\mathrm{Age}(\boldsymbol{M}) = \mathcal{K}$ *and* $\boldsymbol{M}$ *is ultrahomogeneous;*
(2) $\mathrm{Age}(\boldsymbol{M}) = \mathcal{K}$ *and* $\boldsymbol{M}$ *satisfies the* **extension property***, i.e., if* $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$ *and* $f^A\colon \boldsymbol{A} \to \boldsymbol{M}$*,* $f^A_B\colon \boldsymbol{A} \to \boldsymbol{B}$ *are embeddings, then there is* $f^B\colon \boldsymbol{B} \to \boldsymbol{M}$ *so that* $f^A = f^B \circ f^A_B$*;*
(3) $\mathrm{Age}(\boldsymbol{M}) = \mathcal{K}$ *and* $\boldsymbol{M}$ *satisfies the* **1-point extension property** *which is defined as in (2) with the additional restriction* $B \setminus A$.

Proof. First we show that (1),(2) and (3) are equivalent. Property (2) clearly implies (3). Property (3) implies property (1) since it allows us to build a "back and forth system" from $\boldsymbol{M}$ to $\boldsymbol{M}$ starting from any isomorphism $\boldsymbol{A} \to \boldsymbol{A}'$ between finite substructures of $\boldsymbol{M}$. We leave (1) implies (2) as an exercise.

Before we show that such an $\boldsymbol{M}$ satisfying either of (1)–(3) exists notice that uniqueness follows easily since if $\boldsymbol{M}$ and $\boldsymbol{M}'$ satisfy property (3) then it easy to build a "back and forth" system between them.

We show now that if $\mathcal{K}$ is as in the statement of the theorem then there exists $\boldsymbol{M}$ satisfying property (2) above. We will build $\boldsymbol{M}$ as the colimit (union) of a **generic sequence** in $\mathcal{K}$:

$$\boldsymbol{A}_1 \to \boldsymbol{A}_2 \to \boldsymbol{A}_3 \to \cdots \to \boldsymbol{A}_m \to \cdots \to \boldsymbol{A}_n \to \cdots\cdots \quad \boldsymbol{M}$$

That is, a coherent sequence $((\boldsymbol{A}_n), g_n^m)$ of embeddings $g_n^m\colon \boldsymbol{A}_m \to \boldsymbol{A}_n$, $m \leq n$ with $\boldsymbol{A}_n \in \mathcal{K}$ so that

(i) every $\boldsymbol{A}$ from $\mathcal{K}$ embeds eventually in $\boldsymbol{A}_n$ for large n;
(ii) for every $m > 0$ and every embedding $f\colon \boldsymbol{A}_m \to \boldsymbol{B}$ to some $\boldsymbol{B}$ in $\mathcal{K}$ there is $n > m$ and an embedding $f'\colon \boldsymbol{B} \to \boldsymbol{A}_n$ so that $f'f = g_n^m$.

Condition (i) together with the fact that $\boldsymbol{A}_n \in \mathcal{K}$ imply that the age of $\boldsymbol{M} = \bigcup_n \boldsymbol{A}_n$ is indeed $\mathcal{K}$. Property (ii) together with the fact that $\mathcal{K}$ satisfies the amalgamation property implies that $\boldsymbol{M}$ has the extension property from (2).

In order to build a generic sequence in $\mathcal{K}$ we enumerate $(\boldsymbol{B}_n)$ all elements (up to isomorphism) of $\mathcal{K}$. Since $\mathcal{K}$ contains finite structures there are at most finitely many embeddings between any two elements of $\mathcal{K}$. Hence we can also enumerate $(f_n\colon \boldsymbol{C}_n \to \boldsymbol{D}_n)$ all possible embeddings (up to isomorphism) that appear in $\mathcal{K}$ so that each embedding appears infinitely many often in the list.

The construction of $((\boldsymbol{A}_n), g_n^m)$ is now by induction. Let $\boldsymbol{A}_1$ be $\boldsymbol{B}_1$. Assume now that we have built $\boldsymbol{A}_{n-1}$. To build $\boldsymbol{A}_n$ we proceed as follows. First by JEP we can extend $\boldsymbol{A}_{n-1}$ to a structure $\boldsymbol{C}$ so that $\boldsymbol{B}_n$ embeddds in $\boldsymbol{C}$ as well. By finitely many applications of AP enlarge $\boldsymbol{C}$ further to a structure $\boldsymbol{D}$ so that every embedding from $\boldsymbol{C}_n$ to $\boldsymbol{C}$ extends to an embedding from $\boldsymbol{D}_n$ to $\boldsymbol{D}$. Set $\boldsymbol{A}_n$ to be $\boldsymbol{D}$. □

Definition 32. The structure $\boldsymbol{M}$ associated to a Fraïssé class $\mathcal{K}$ as in the statement of Theorem 31 is called the **Fraïssé limit** of $\mathcal{K}$.

Since Theorem 22 produces ultrahomogeneous structures we have, at least abstractly, many examples of Fraïssé structures. Here is a collection of more natural examples which will be important later on.

Example 33.
Let $\mathcal{K}_{\mathrm{LO}}$ be the collection of all finite linear orders. It is easy to see that $\mathcal{K}_{\mathrm{LO}}$ is a Fraïssé class. A way to see this is to recall that $(\mathbb{Q}, \leq)$ is ultrahomogeneous and its age is $\mathcal{K}_{\mathrm{LO}}$. In other words, the Fraïssé limit of $\mathcal{K}_{\mathrm{LO}}$ is $(\mathbb{Q}, \leq)$.
Let $\mathcal{K}_{\mathrm{Graphs}}$ be the collection of all finite graphs. It is easy to see that $\mathcal{K}_{\mathrm{Graphs}}$ is a Fraïssé class. It's limit is known as the **random graph** or **Rado graph**. While the most useful characterization of the Random graph is via the extension property (see Theorem 31) we point out that the Random graph can be defined as the isomorphism type one attains with probability 1 if you put an edge between any pair of a countable set of vertexes with probability 1/2.
Similarly, for every $n \geq 3$ the collection$\mathcal{K}^n_{\mathrm{Henson}}$ of all finite graphs which do not have a clique of size n forms a Fraïssé class. The Fraïssé limit of this class is known as the K_n-free random graph or as the n-Henson graph.
Consider the collection $\mathcal{K}_{\mathbb{QU}}$ of all finite metric spaces with rational distances. We can view these metric spaces as $\mathcal{L}$-structures where $\mathcal{L} = \{d_q \mid q \in \mathbb{Q}^+\}$ is the language that contains a binary predicate for every positive rational. Again, $\mathcal{K}_{\mathbb{QU}}$ is a Fraïssé class whose limit is known as the rational Urysohn space $\mathbb{QU}$. The completion of $\mathbb{QU}$ is a very important metric spaces known as the universal Urysohn space $\mathbb{U}$.

If $\mathcal{K}$ is a Fraïssé class then we denote by $\mathcal{K}^\omega$ the collection of all countable structures $\boldsymbol{N}$ whose age is included in $\mathcal{K}^\omega$. These are precisely all colimits of a countable coherent direct system of embeddings from $\mathcal{K}$. The sequence (A_n, g_n^m) in the proof of the Theorem 31 is called generic because the properties (i),(ii) which define it are generic in a certain sense. Informally think of a game played with two players, Player I and Player II. Player I starts and plays $\boldsymbol{A}_1 \in \mathcal{K}$ then Player II replies by playing $\boldsymbol{A}_2 \in \mathcal{K}$ together with an embedding $g_2^1 \colon \boldsymbol{A}_1 \to \boldsymbol{A}_2$. Then Player I plays $\boldsymbol{A}_3 \in \mathcal{K}$ together with an embedding $g_3^2 \colon \boldsymbol{A}_2 \to \boldsymbol{A}_3$, and the game continues similarly by alternating between Player I and Player II. Properties (i) and (ii) in the proof of Theorem 31 are called generic because if the task of Player II is to enforce them to the growing sequence then then no matter what Player I plays Player II has a winning strategy for doing so. A more formal way to put it is as follows. If the structures of $\mathcal{K}$ are $\mathcal{L}$-structures with $\mathcal{L}$ being a countable relational language then the collection $\mathcal{X}_\mathcal{L}$ of all $\mathcal{L}$-structures on domain $\mathbb{N}$ can be given a Polish topology in the form of the following product of Cantor sets:

$$\mathcal{X}_\mathcal{L} = \Pi_{R\in\mathcal{L}} \;\; 2^{\mathbb{N}^{\alpha(R)}}$$

Exercise. Show that the space $\mathcal{K}^\omega \cap \mathcal{X}_\mathcal{L}$ is just a closed subset of $\mathcal{X}_\mathcal{L}$. Show that the collection of all structures of $\mathcal{K}^\omega \cap \mathcal{X}_\mathcal{L}$ which are isomorphic to the Fraïssé limit of $\mathcal{K}$ are forms a dense G_δ subset in $\mathcal{K}^\omega \cap \mathcal{X}_\mathcal{L}$.

Exercise. Show that every $\boldsymbol{N} \in \mathcal{K}^\omega$ embeds in the Fraïssé limit of $\mathcal{K}$.

Exercise. Show that the Urysohn space $\mathbb{U}$ is a separable complete metric space which contains all finite (or even separable) metric spaces. Moreover it satisfies the extension property with respect to embeddings between finite metric spaces. Show that $\mathbb{U}$ is ultrahomogeneous.

5. Week 5

The correspondence between Fraïssé classes $\mathcal{K}$ and ultrahomogenous structures $\boldsymbol{M}$ give by Theorem 31 and Lemma 29 allows us to build a dictionary between

(1) combinatorial properties between structures of $\mathcal{K}$;
(2) dynamical properties of closed subgroups of S_∞.

This correspondence extends within the more general framework of "metric Fraïssé theory" to a correspondence between certain classes of finite metric structures and of general Polish groups. Here we will stay in the realm of discrete structures since the ideas we want to present are more transparent in this context. That being said we will still be interested in canonical examples in metric context such as the Urysohn space $\mathbb{U}$. We summarize here two equivalent definitions of $\mathbb{U}$.

Definition 34. The **Urysohn space** is the unique separable metric space $\mathbb{U}$ satisfying either of the following equivalent statements with the properties

(1) every finite metric space A isometrically embeds in $\mathbb{U}$ and $\mathbb{U}$ is ultrahomogeneous (with respect to all isometries between finite subspaces);
(2) every finite metric space A isometrically embeds in $\mathbb{U}$ and $\mathbb{U}$ has the one point extension property (with respect to all pairs $A \subseteq B$ of finite metric spaces with $|B/A| = 1$)

We can use $\mathbb{U}$ to prove the following theorem.

Theorem 35. *There exists a universal Polish group, i.e., a Polish group G so that every other Polish group embeds in G as a closed subgroup.*

We will need some definitions first. Let (X, d) be a separable metric space. A **Katetov map over** X is any map $f\colon X \to \mathbb{R}$ so that

$$\forall x, y \in X \quad |f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y).$$

Notice that Katetov maps correspond precisely to one point extensions $X \rightsquigarrow X \cup \{*_f\}$ of the metric space X where $d(x, *_f)$ is given by $f(x)$. A Katetov map has **finite support**, if there is a finite subset S of X so that

$$f(x) = \inf_{s \in S} (f(s) + d(x, s))$$

Let $E(X)$ be the collection of all finitely supported Katetov maps. We view $E(X)$ as a metric space with the supremum distance

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)| = \sup_{x \in \mathrm{Supp}(f)} |f(x) - g(x)| = \sup_{x \in \mathrm{Supp}(g)} |f(x) - g(x)|$$

which we still denote by d in view of the following lemma.

Lemma 36. *$E(X)$ is a separable metric space and for every $x \in X$ the function $\delta_x\colon X \to \mathrm{R}$ given by $\delta_x(y) = d(x, y)$ is in $E(X)$. Moreover, the map*

$$x \mapsto \delta_x$$

Is an embedding of X to $E(X)$ so that every isometry of X extends uniquely to an isometry of $E(X)$.

Proof. Easy and left to the reader. □

We can now form the space $\overline{E^\infty(X)}$ which is the completion of the direct system

$$X \hookrightarrow E(X) \hookrightarrow E(E(X)) \hookrightarrow \cdots$$

Lemma 37. *The space $\overline{E^\infty(X)}$ is isometric to the Urysohn space $\mathbb{U}$.*

Proof. $\overline{E^\infty(X)}$ is clearly complete separable. It is also easy to see inductively that the n-th iterate $E(\ldots E(X)\ldots)$ contains every finite metric space of size n. So we are left to show that $\overline{E^\infty(X)}$ satisfies the 1-point extension property. For that let $x^1,\ldots,x^n$ be a finite subset of $\overline{E^\infty(X)}$ if $\{x^1,\ldots,x^n\} \subset E^\infty(X)$ then $\{x^1,\ldots,x^n\} \subset E^n(X)$ for some n and $E^{n+1}(X)$ contains all 1-point extensions of $\{x^1,\ldots,x^n\}$.

Otherwise, if $\{x^1,\ldots,x^n\} \not\subset E^\infty(X)$ then let $(q^i_k)_k$ be a sequence in $E^\infty(X)$ converging to x^i. Then any 1-point extension over $\{x^1,\ldots,x^n\}$ can be realized in $\overline{E^\infty(X)}$ by using inductively the previous case to construct a Cauchy sequence (p_k) in $E^\infty(X)$ so that the space $\{q^1_k,\ldots,x^n_k,p_k\}$ converges to the required extension. □

Proof of Theorem 35. We will show that $\mathrm{Iso}(\mathbb{U})$ is universal. Let G be a Polish group. By 10 we can pick a left invariant metric d for G. Notice that G is a closed subgroup of the Polish group $\mathrm{Iso}(X,d)$, where X is the left completion of G. By Lemma 36 $\mathrm{Iso}(X,d)$ embeds in $E(X)$ and therefore in $E(E(X))$, etc. Following the arrows and taking the completion we have be Lemma 37 an embedding of $\mathrm{Iso}(X,d)$ to $\mathrm{Iso}(\mathbb{U})$. The rest follows by Lemma 3. □

Exercise. Show that the automorphism group of the random graph is universal for all Polish groups G which are isomorphic to automorphism groups of graphs.

Next we elaborate on the a simple but very important correspondence between Fraïssé classes and dynamical properties of the associated automorphism group.

Let $\mathcal{K}$ be a Fraïssé class. We say that $\mathcal{K}$ has the **Hrushovski property** the ultrahomogeneity of the Fraïssé limit of $\mathcal{K}$ can be "approximated by the finite structures," i.e., if for every $\boldsymbol{A} \in \mathcal{K}$ there exists an embeddings $i\colon \boldsymbol{A} \to \boldsymbol{B}$ in $\mathcal{K}$ so that for every partial isomorphism p of $\boldsymbol{A}$ in $\boldsymbol{B}$ to an automorphism of $\boldsymbol{B}$.

Example. if $\boldsymbol{A}$ is the graph $\circ - \circ - \circ$ the 4-point cycle graph can be used as the Hrushovski extension of $\boldsymbol{A}$. We will see that more generally the class of graphs has the Hrushovski property. From the other hand notice that the class of finite linear orders does not have this property since all structures there are rigid.

Theorem 38. *Let $\mathcal{K}$ be a Fraïssé class and let $\boldsymbol{M}$ be its Fraïssé limit. The following are equivalent:*

(1) $\mathcal{K}$ has the Hrushovski property;
(2) $\mathrm{Aut}(\boldsymbol{M})$ *is* **compactly approximable**, *i.e. there exists an increasing sequence of compact subgroups whose union is dense in* $\mathrm{Aut}(\boldsymbol{M})$.

Proof. If $\mathcal{K}$ has the Hrushovski property then when we were building the generic sequence $((\boldsymbol{A}_n), g_n^m)$ in Theorem 31 we could have arranged that $\boldsymbol{A}_{n+1}$ is a Hrushovski extension of $\boldsymbol{A}_n$.

For every fixed $N > 0$ and any $n > N$ we can consider the subgroup H_n^N of $\mathrm{Aut}(\boldsymbol{A}_n)$ consisting of all automorphisms of $\boldsymbol{A}_n$ which setwise stabilize the image of A_m under g_n^m in A_n, for all m in the interval $[N, n)$. As a consequence we have a homomorphism $\phi_m^n \colon H_n^N \to H_m^N$ whenever $m \leq n$ and the Hrushovski property implies that this is an epimorphism.

Let G_N be the inverse limit of the inverse system (H_n^N, ϕ_m^n). It is immediate that G_N is compact and that for all $N > 0$ there is a canonical embedding $G_N \hookrightarrow G_{N+1}$. Hence (G_N) is an increasing sequence of finite subgroups of $\mathrm{Aut}(\boldsymbol{M})$ which is moreover dense: if V_p is the basic open in $\mathrm{Aut}(\boldsymbol{M})$ determined by the finite partial isomorphism, then both the domain and the range of p are included in $\boldsymbol{A}_{N-1}$ for large enough N and therefore there exists $g \in \mathrm{Aut}(\boldsymbol{A}_N)$ so that $g \restriction \mathrm{dom}(p) = p$. Since the inverse system (H_n^N, ϕ_m^n) consists of epimorphisms, we can lift g to an element $\tilde{g} \in G_N < \mathrm{Aut}(\boldsymbol{M})$ and as a consequence $G_N \cap V_p \neq \emptyset$.

Conversely, let $\boldsymbol{A} \leq \boldsymbol{M}$ and set $\mathcal{P}(\boldsymbol{A})$ be the collection of all partial isomorphism of $\boldsymbol{A}$. By density of the increasing sequence of the compact subgroups we can find a compact subgroup K of $\mathrm{Aut}(\boldsymbol{M})$ so that every $p \in \mathcal{P}(\boldsymbol{A})$ extends to an element in K. Since the action of $\mathrm{Aut}(\boldsymbol{M})$ on M is continuous when M is endowed with the discrete topology, we have that the orbit closure of the finite A under the compact K is a compact and therefore a finite subset of M. But then $\boldsymbol{B} = \langle K \cdot A \rangle_M$ is a Hrushovski extension of $\boldsymbol{A}$. □

Remark 39.
1. Since the class of all finite linear orders does not have the Hrushovski property, we have that $(\mathbb{Q}, \leq)$ is not compactly approximable.
2. Since compact topological groups are amenable, and since every topological group that contains an increasing sequence of amenable subgroups with dense union is amenable, if $\mathcal{K}$ has the Hrushovski property then the automorphism group of the Fraïssé limit of $\mathcal{K}$ is amenable.

Theorem 40 (Hrushovski)**.** *The class class of finite graphs has the Hrushovski property.*

Proof. For every finite set X and every natural number $n \leq |X|$ let $G(X, k)$ be the graph whose vertexes are all k-subsets of X and an edge between such k-subsets S and S' exists if and only if $S \cap S' \neq \emptyset$.

A subgraph A of $G(X, k)$ is **thin** if for every two vertexes S and S' of A we have that $|S \cap S'| \leq 1$ and every $x \in X$ appears in at most vertexes of A. In other words, there is a well defined injection from edges of A to X. Notice that every permutation $\sigma \in S_X$ induces an automorphism $\tilde{\sigma}$ of $G(X, k)$.

Claim. *Every isomorphism $p \colon A \to B$ between thin subgraphs of $G(X, k)$ extends to an automorphism of $G(X, k)$.*

Proof of Claim. It is enough to construct $\sigma \in S_X$ so that $\tilde{\sigma}$ extends σ. Notice that A partitions X to the sets X_0^A, X_1^A, X_2^A with X_i^A being the collection of all x which lie in exactly i-many vertexes of A. Similarly X is partitioned by B to the sets X_0^B, X_1^B, X_2^B. Since A, B are isomorphic and thin we have that $|X_i^A| = |X_i^B|$. Let σ be any permutation of X that sends X_0^A to X_0^B and X_1^A to X_1^B in arbitrary fashion, and maps X_2^A to X_2^B in the obvious way: if S, S are the two distinct vertexes of A with $x \in S \cap S'$ then set $\sigma(x)$ to be the unique element of $p(S) \cap p(S')$. □

Claim. *For every graph A there is a finite X and k so that A embeds in $G(X, k)$ as a thin subgraph.*

Proof of Claim. We can assume without loss of generality that each vertex of A has degree k for some fixed large enough k. To see this, take k to be the smallest odd number larger than maxdegree(A). Extend A to A' by adding to each $v \in A$ as many new edges (intersecting with A only on v) so that each vertex in A has degree k in A'. All new vertexes, i.e., all vertexes in $A' \setminus A$ have degree 1. If $|A' \setminus A| \geq k-1$ we can cyclically order $A' \setminus A$ them and connect each $v \in A' \setminus A$ with its $(k-1)/2$ predecessors and its $(k-1)/2$ successors. The resulting graph A'' will have uniform degree k. Finally the case $|A' \setminus A| < k-1$ reduces to the case $|A' \setminus A| \geq k-1$ by simply adding in the very beginning of the process a new vertex to A isolated from the others. Then the resulting A' would have the property $|A' \setminus A| \geq k-1$.

Let now A be a graph of degree k and let X be the set of all edges of A. Then the map sending each vertex v of A to the set of all edges in A that contain v is the desired embedding of A as a thin subgraph of $G(X, k)$. □

Combining the above two claims and observing that a subgraph of a thin graph in $G(X, k)$ is thin we have that the class of all graphs has the Hrushovski property. □

The above proof is somewhat specific to the case of graphs. Herwig and Lascar developed a very general technique for proving the Hrushovski property which applies in many other cases. This technique makes use of the following theorem of Hall which allows one to generalize the way we attained the Hrushovski extension of the graph $\circ - \circ - \circ$ in the example before Theorem 40.

Theorem 41. *Every finitely generated subgroup of the free group F_n is closed in the profinite topology of F_n.*

Problems. Here are two problems which are open currently:

1. A tournament is a directed graph with the property that for every two vertexes x, y either $(xRy) \wedge \neg(yRx)$ or $(yRx) \wedge \neg(xRy)$. The class of all finite tournaments is a Fraïssé class. It remains open whether it satisfies the Hrushovski property. Some partial results have been attained recently in a joint paper of (J. Huang, M. Pawliuk, M. Sabok, D. Wise)
2. Let $\mathcal{K}_K$ be the class of all compact metric spaces. In a joint work of S. Solecki and M. Etedadialiabadi it is shown that $\mathcal{K}_K$ satisfies an approximate version of the Hrushovski property, i.e., if A is a compact metric space then for every

$\varepsilon > 0$ there exists an isometric embedding $i\colon A \to B$ to a compact metric space so that for every partial isometry p of A there is $\varphi \in \mathrm{Iso}(B)$ so that $d(\varphi \circ i(x), i \circ p(x)) < \varepsilon$ for every $x \in \mathrm{dom}(p)$. It is unclear whether one can prove the same thing for $\varepsilon = 0$.

The next theorem is a splendid illustration of how Fraïssé methods provide a bridge between finite combinatorics and dynamics of Polish groups. We will now state and elaborate one the relevant notions. The proof will be broken up into several lemmas, some of them interesting on their own right.

Theorem 42 (Kechris-Pestov-Todorcevic)**.** *Let G be a closed subgroup of S_∞. Then the following are equivalent:*

(1) G is extremely amenable;
(2) G is the automorphism group of the Fraïssé limit of a class $\mathcal{K}$ which has the Ramsey property.

Let us first define the terms involved in the above statement.

Let G be a Polish group. A **compact G-space** is a compact topological space K together with a continuous action of G on K, i.e., a continuous map $\alpha\colon G \times K \to K$ so that $\alpha(1, x) = x$ and $\alpha(gh, x) = \alpha(g, \alpha(h, x))$ for all $x \in K$ and $g, h \in G$. We will use the notation $g \cdot x$ or often just gx for $\alpha(g, x)$. A Polish group G is **extremely amenable** if for every compact G-space has a fixed point, i.e. a $x \in K$ so that $gx = x$ for all $g \in G$.

Given a pair of structures $\boldsymbol{A}, \boldsymbol{B}$ in $\mathcal{K}$, with $\boldsymbol{A} \leq \boldsymbol{B}$, we denote by $\binom{\boldsymbol{B}}{\boldsymbol{A}}$ the collection of all embeddings $f\colon \boldsymbol{A} \to \boldsymbol{B}$. Given $\boldsymbol{A} \leq \boldsymbol{B} \leq \boldsymbol{C}$ in $\mathcal{K}$ and a number (of colors) $k \geq 2$ we write

$$\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_k$$

if for every coloring $r\colon \binom{\boldsymbol{C}}{\boldsymbol{A}} \to \{1, \dots, k\}$, of the collection of all copies of $\boldsymbol{A}$ in $\boldsymbol{C}$ in k-many colors, there exists a copy $g\colon \boldsymbol{B} \to \boldsymbol{C}$ of $\boldsymbol{B}$ in $\boldsymbol{C}$ so that

$$g \circ \binom{\boldsymbol{B}}{\boldsymbol{A}} := \{g \circ f \colon f \in \binom{\boldsymbol{B}}{\boldsymbol{A}}\} \subseteq r^{-1}(i),$$

for some $i \in \{1, \dots, k\}$, i.e., the set of all copies of $\boldsymbol{A}$ in $\boldsymbol{C}$ which lie in the fixed "copy g" of $\boldsymbol{B}$ in $\boldsymbol{C}$ is monochromatic. The Fraïssé class $\mathcal{K}$ has the **Ramsey property** if for every pair $\boldsymbol{A} \leq \boldsymbol{B}$ in $\mathcal{K}$ and every number $k \geq 2$ there is $\boldsymbol{C}$ in $\mathcal{K}$ with $\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_k$.

Remark 43. Extreme amenability is a very strong property. Notice for example that whenever $\mathrm{Aut}(\boldsymbol{M})$ is extremely amenable then $\boldsymbol{M}$ has to carry implicitly or explicitly some ordering. If the ordering is implicit we can reveal it using that $\mathrm{Aut}(\boldsymbol{M})$ is extremely amenable: let M be the domain of $\boldsymbol{M}$ and consider the space $\mathrm{LO}(M)$ be the space of all linear orderings on M. Since $\mathrm{LO}(M)$ is a closed subset of $\{0, 1\}^{M \times M}$ the space $\mathrm{LO}(M)$ is compact. Moreover, there is a natural action of $\mathrm{Aut}(\boldsymbol{M})$ on $\mathrm{LO}(M)$ given by

$$m \ (g \cdot \leq) \ m' = g^{-1}(m) \leq g^{-1}(m'),$$

for all $\leq \in \mathrm{LO}(M)$. Since $\mathrm{Aut}(\boldsymbol{M})$ is extremely amenable, there exists some linear ordering $\leq$ on M that is preserved under automorphisms of $\boldsymbol{M}$ and hence we can add it as part of the structure without any cost on the symmetries of $\boldsymbol{M}$.

Let's prove the classical Ramsey theorem disguised in the following form.

Theorem 44 (Ramsey)**.** *The class $\mathcal{K}_{LO}$ of all finite linear orders has the Ramsey property.*

Proof. We need to show that for every positive natural numbers $a \leq b$ and any number of colors $k \geq 2$ there exists a large enough $b \leq c$ so that

$$c \to (b)^a_k.$$

The last expression stands for: "for any coloring of the collection $\binom{c}{a}$ of all a-sized subsets of $\{1, 2, \ldots, c\}$ there exists a b-sized subset of $\{1, 2, \ldots, c\}$, all of whose a-sized subsets have been colored in the same color."

It will be convenient to prove an infinitary version of this usual Ramsey statement and use compactness in the form of König's lemma to derive the above finitary version.

Claim. *For every natural number $a \geq 1$ and any number of colors $k \geq 2$ we have*

$$\omega \to (\omega)^a_k.$$

To see how this infinitary statement implies the finite one assume towards contradiction that for some $a \leq b$ and $k \geq 2$ there is no c with $c \to (b)^a_k$. We say that a k-coloring of $\binom{c}{a}$ is **bad** if it that witness that $c \not\to (b)^a_k$. Notice that if $c' < c$ then any coloring r of $\binom{c}{a}$ induces a coloring $r' = r\restriction\binom{c'}{a}$ of $\binom{c'}{a}$ and that if r is bad so is r'. As a consequence, if $\mathrm{BadCol}(c)$ is the collection of all bad colorings of$\binom{c}{a}$ then the set $\bigcup_{c\in\mathbb{N}} \mathrm{BadCol}(c)$ can be viewed as a finitely branching tree where where r extends r' if $r' = r\restriction\binom{c'}{a}$. Since we assumed that there is no c with $c \to (b)^a_k$ we have that this tree is infinite. By Königs lemma we have an infinite branch of colorings whose union would provide a bad coloring of $\binom{\omega}{a}$ contradicting the claim.

Proof of claim. First notice that for every fixed a the case for the general k follows from the one for $k = 2$. For example, consider the case $k = 3$ and let $r\colon \binom{\omega}{a} \to \{\mathrm{red}, \mathrm{blue}, \mathrm{yellow}\}$ be a three coloring of a-subsets of ω. Combining blue and yellow to green we get an induced 2-coloring $r'\colon \binom{\omega}{a} \to \{\mathrm{red}, \mathrm{green}\}$ where $r'(A) = \mathrm{red}$ if $r(A) = \mathrm{red}$ and $r'(A) = \mathrm{green}$ otherwise. If $\omega \to (\omega)^a_2$ then we can find an infinite X of ω so that $\binom{X}{a}$ is monochromatic with respect to r'. If $r'\restriction\binom{X}{a} = \mathrm{red}$ we are done since then $\binom{X}{a}$ is monochromatic with respect to r. Otherwise $r'\restriction\binom{X}{a} = \mathrm{green}$ and therefore the restriction of the original coloring r on $\binom{X}{a}$ is a two coloring with colors $\{\mathrm{blue}, \mathrm{yellow}\}$ which reduces again to the $\omega \to (\omega)^a_2$. The same idea generalizes to an inductive argument for the general k.

Therefore it suffice to prove that for every fixed a we have that $\omega \to (\omega)^a_2$. For $a = 1$ this is simply the pigeonhole principle: for every partition of ω into two pieces one of the pieces has to be infinite. We illustrate now how to prove it for $a = 2$ (given the case $a = 1$). It is a good exercise for the reader to modify this argument and prove the general inductive step a (given that it works for $a - 1$).

Fix any coloring of $\binom{\omega}{2}$ with colors $\{\text{red}, \text{blue}\}$. We first will use pigeonhole principle infinitely many times to get an infinite subset X of ω so that the color of each element $\{x < x'\}$ of $\binom{X}{2}$ depends on a and then use the inductive hypothesis to determine an infinite subset $\tilde{X}$ of X so that $\binom{\tilde{X}}{2}$ is monochromatic.

Defining X. Let $X_0 = \omega$ and let $x_0 = \min X_0 = 0$ by pigeonhole principle there is an infinite $X_1 \subseteq X_0$ so that $\{(x_0, x) \mid x \in X_1\}$ is monochromatic. Let now $x_1 = \min X_1$. By pigeonhole principle there is an infinite $X_2 \subseteq X_1$ so that $\{(x_1, x) \mid x \in X_2\}$ is monochromatic. We continue inductive by picking $x_{n-1} = \min X_{n-1}$ and setting X_n to be an infinite subset an infinite subset of X_{n-1} so that $\{(x_{n-1}, x) \mid x \in X_n\}$ is monochromatic. Let $X = \{x_0, x_1, \ldots\}$.

Defining $\tilde{X}$. Since the coloring $r\colon \binom{X}{2} \to \{\text{red}, \text{blue}\}$ depends only on the "first" coordinate of each $\{x < x'\} \in \binom{X}{2}$, r induces a coloring r' on $\binom{X}{1} = X$ given by $r'(x) = r(\{x < x'\})$, for any $x' > x$. By the inductive step ($a = 1$) there is an infinite subset $\tilde{X}$ of X so that $r' \restriction \binom{X}{1}$, and therefore $r \restriction \binom{X}{2}$, is monochromatic.

□

□

6. Week 6

In the process of proving 42 we will develop some theory around the notion of *finite oscillation stability*. This, as we are going to see, is a dynamical version of Ramsey theory and it is best formulated in the context of *uniform G-spaces*.

Definition 45. A **uniform space** is a pair $(X, \mathcal{U})$ so that X is a set and $\mathcal{U}$ is is a collection of subsets of $X \times X$ called **entourages**, so that:

(1) every $\mathcal{E} \in \mathcal{U}$ contains the diagonal $\Delta_X = \{(x, x) \mid x \in X\}$;
(2) if $\mathcal{E} \in \mathcal{U}$ then $\mathcal{E}^{-1} = \{(y, x) \mid (x, y) \in \mathcal{E}\} \in \mathcal{U}$;
(3) $\mathcal{U}$ is closed under finite intersections and supersets, i.e., if $\mathcal{E} \in \mathcal{U}$ and $\mathcal{E}' \supseteq \mathcal{E}$ then $\mathcal{E}' \in \mathcal{U}$;
(4) if $\mathcal{E} \in \mathcal{U}$ then there exists some $\mathcal{E}' \in \mathcal{U}$ so that $\mathcal{E}' \circ \mathcal{E}' \subseteq \mathcal{E}$ where

$$\mathcal{E}' \circ \mathcal{E}' = \{(x, z) \mid \exists y \in X (x, y) \in \mathcal{E}', (y, z) \in \mathcal{E}'\}.$$

If $(X, \mathcal{U})$ is a uniform space, $\mathcal{E}$ is an entourage, and $A \subseteq X$ then we denote by $\mathcal{E}[A]$ the "$\mathcal{E}$-blow up" of A, i.e., the set $\{x \in X \mid \exists y \in A (x, y) \in A\}$. A uniform space $(X, \mathcal{U})$ induces always a topology whose neighborhoods are $\{\mathcal{E}[x] \mid x \in X, \mathcal{E} \in \mathcal{U}\}$. Similarly a metric space (X, d) has a canonical uniformity $(X, \mathcal{U}_d)$ whose entourages are all supersets of sets of the form $\{(x, y) \mid d(x, y) < \varepsilon\}$. The same holds for **pseudometrics**, i.e., functions $d\colon X \times X \to \mathbb{R}$ which satisfy all axioms of a metric except that $d(x, y) = 0$ does not imply $x = y$. In the same way topology and metric spaces provide the right frameworks for studying continuity and isometries, uniform spaces provide the right framework for studying uniformly continuous functions.

Definition 46. A function $f\colon X \to Y$ between the uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is uniformly continuous if for every $\mathcal{E} \in \mathcal{V}$ there is $\mathcal{D} \in \mathcal{U}$ so that $(f \times f)(\mathcal{D}) \subseteq \mathcal{E}$.

A pseudometric d on $(X,\mathcal{U})$ is **uniformly continuous** if the identity map $(X,\mathcal{U}) \to (X,\mathcal{U}_d)$ is uniformly continuous. For every uniformity $(X,\mathcal{U})$ there exists a family $\mathcal{R}$ of uniformly continuous pseudometrics on $(X,\mathcal{U})$ which generate $\mathcal{U}$, in that, for every $\mathcal{E} \in \mathcal{E}$ there exists $\rho \in \mathcal{R}$ and $\varepsilon > 0$ so that $\{(x,y) \mid \rho(x,y) < 1\} \subseteq \mathcal{E}$. In fact we can always take $\mathcal{R}$ to be **directed**, i.e., if $\rho_1, \rho_2 \in \mathcal{R}$ then there exists $\rho \in \mathcal{R}$ so that $\rho_1, \rho_2 \leq \rho$ pointwise.

Many constructions notions and constructions usually associated with metric spaces apply in the context of uniform spaces, e.g, boundedness, completion, etc. Moreover many constructions relevant to topological spaces "factor through" a uniformity. Such is the process of forming a compactification.

A uniform space $(X,\mathcal{U})$ is **totally bounded** if for every $\mathcal{E} \in \mathcal{U}$ there exists a finite set F so that $X = \mathcal{E}[F]$. The **totally bounded replica** $(X,\mathcal{C}^*\mathcal{U})$ of a uniform space $(X,\mathcal{U})$ is the the finest uniformity contained in $\mathcal{U}$ that is totally bounded. The completion of $(X,\mathcal{C}^*\mathcal{U})$ is denoted by $\mathcal{S}(X,\mathcal{U})$ and it is the called the **Samuel compactification** of $(X,\mathcal{U})$. It is the maximal ideal space of the commutative C^*-algebra $\mathrm{UC}_b(X)$ of all uniformly continuous bounded complex valued functions on $(X,\mathcal{U})$. In other words, σX consists of all multiplicative continuous linear functionals on the Banach space $\mathrm{UC}_b(X)$ endowed with the weak*-topology.

Exercise 47.
1. *If $(X,\mathcal{U})$ is a uniform space then every entourage $\mathcal{E}$ is a neighborhood of the diagonal Δ_X in the product topology on $X \times X$.*
2. *If X is a compact topological space then there exists a unique uniformity $\mathcal{U}$ compatible with the topology of X. That is the one generated by all open neighborhoods of the diagonal in $X \times X$.*

Let G be a Polish group. A **uniform G-space** is a uniform space $(X,\mathcal{U})$ together with an action of G on X by uniform isomorphisms, i.e. $\alpha(g,\cdot)\colon X \to X$ is a uniformly continuous bijection for all $g \in G$.

Definition 48. Let G be a Polish group and let $(X,\mathcal{U})$ be a uniform G-space. A function f on X, taking values on a uniform space $(Y,\mathcal{V})$ is **finitely oscillation stable** if for every finite $F \subseteq X$ and every $\mathcal{E} \in \mathcal{V}$ there is $g \in G$ so that $f(gF)$ is $\mathcal{E}$-small, i.e.,$f(gF) \times f(gF) \subseteq \mathcal{E}$.

We say that $(X,\mathcal{U})$ is **finitely oscillation stable** if for every bounded uniformly continuous $f\colon (X,\mathcal{U}) \to \mathbb{R}$ is finitely oscillation stable.

The following theorem illustrates how finite oscillation stability can be seen as a dynamical version of the Ramsey property.

Theorem 49. *Let $(X,\mathcal{U})$ be a uniform G-space. Then the following are equivalent.*

(1) $(X,\mathcal{U})$ *is finitely oscillation stable.*
(2) *If* $\mathcal{R}$ *is a directed collection of bounded uniformly continuous pseudometrics generating the uniform structure of* $(X,\mathcal{U})$ *and* $\rho \in \mathcal{R}$ *then any* 1*-Lipschitz real valued function on* (X,ρ) *is finitely oscillation stable.*
(3) *For every binary cover* $\{A,B\}$ *of* X*, every* $\mathcal{E} \in \mathcal{U}$*, and every finite* $F \subseteq X$ *there exists* $g \in G$ *so that* $gF \subseteq \mathcal{E}[A]$ *or* $gF \subseteq \mathcal{E}[B]$*.*
(4) *For every* $\mathcal{E} \in \mathcal{U}$ *and every finite* $F \subseteq X$*, there is a finite* $K \subseteq X$ *so that for every binary cover* $\{A,B\}$ *of* K *there exists* $g \in G$ *so that* $gF \subseteq \mathcal{E}[A]$ *or* $gF \subseteq \mathcal{E}[B]$*.*
(5) *For every* $\mathcal{E} \in \mathcal{U}$*, every finite* $F \subseteq X$ *and every* $l \geq 1$*, there is a finite* $K \subseteq X$ *so that for every cover* $\{A_1,\dots,A_l\}$ *of* K *there exists* $g \in G$ *and* $i \leq l$ *so that* $gF \subseteq \mathcal{E}[A_i]$*.*
(6) *For every finite cover* $\mathcal{A}$ *of* X*, every* $\mathcal{E} \in \mathcal{U}$*, and every finite* $F \subseteq X$ *there exists* $g \in G$ *so that* $gF \subseteq \mathcal{E}[A]$ *for some* $A \in \mathcal{A}$*.*
(7) *The embedding* $i\colon X \to \mathcal{S}(X)$ *to the Samuel compactification is finite oscillation stable.*
(8) *If* Y *is any compact space and* $f\colon X \to Y$ *is uniformly continuous then* f *is finitely oscillation stable.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3): since $\mathcal{R}$ generates $\mathcal{U}$ we can find a pseudometric $\rho \in \mathcal{R}$ and $\varepsilon > 0$ so that $d(x,y) < \varepsilon$ implies $(x,y) \in \mathcal{E}$. Consider now the map $f\colon X \to \mathbb{R}$ with $f(x) = \rho(x,A)$. Since f is clearly 1-Lipschitz, (2) implies that there is g so that $\mathrm{diam}_{\mathbb{R}}(f(gF)) < \varepsilon$, and therefore $\mathrm{diam}_{\rho}(gF) < \varepsilon$. If $gF \subseteq B$ we are done. Otherwise $gF \cap A \neq \emptyset$ and therefore $gF \subseteq \mathcal{E}[A]$.

(3) $\Rightarrow$ (4): assume (4) does not hold, i.e., there exists some $\mathcal{E} \in \mathcal{U}$ and some finite $F \subseteq X$ so that for every finite $K \subseteq X$ there exists a **bad cover** $\{A_K, B_K\}$ of K, i.e., a cover so that for every $g \in G$ we have that $gF \not\subseteq \mathcal{E}[A]$ and $gF \not\subseteq \mathcal{E}[B]$.

Since the collection $\mathcal{P}_{\mathrm{fin}}(X)$ of all finite subsets of X is directed, we can form a non-principle ultrafilter ξ on $\mathcal{P}_{\mathrm{fin}}(X)$, i.e., an ultrafilter containing all subsets of $\mathcal{P}_{\mathrm{fin}}(X)$ of the form $\{K \in \mathcal{P}_{\mathrm{fin}}(X) \mid K \supseteq L\}$, with $L \in \mathcal{P}_{\mathrm{fin}}(X)$. This ultrafilter will allow us to define a "bad cover" of X, contradicting (3).

Let $A = \lim_{K\to\xi} A_K$ which contains all $x \in X$ with $\{K \in \mathcal{P}_{\mathrm{fin}}(X) \mid a \in A_K\} \in \xi$. Similarly define $B = \lim_{K\to\xi} B_K$. Since for every $x \in X$ we have that $\{K \in \mathcal{P}_{\mathrm{fin}}(X) \mid x \in K\}$ is in ξ and since ξ is a non-principle (contains $\{K \in \mathcal{P}_{\mathrm{fin}}(X) \mid K \supseteq L\}$ for all $L \in \mathcal{P}_{\mathrm{fin}}(X)$) ultrafilter we have that $\{A,B\}$ covers X. We claim that this cover is "bad', i.e., for any fixed g, we have $gF \not\subseteq \mathcal{E}[A]$ and $gF \not\subseteq \mathcal{E}[B]$. Fix therefore any $g \in G$. Since for K we have that $\{A_K, B_K\}$ is a bad cover we can pick $a_K, b_K \in gF$ so that a_K is not in $\mathcal{E}[A_K]$ and b_K is not in $\mathcal{E}[B_K]$. But gF is a finite set and $\{K \mid K \supseteq gF\}$ is in ξ. Therefore there exists $a = \lim_{K\to\xi} a_K \in gF$ and $b = \lim_{K\to\xi} b_K \in gF$. It is easy to see that a is not in $\mathcal{E}[A]$ and b is not in $\mathcal{E}[B]$.

(4) $\Rightarrow$ (5): it is a standard induction as in the proof of Claim 5.

(5) $\Rightarrow$ (6): obvious.

(6) ⇒ (7): it is easy to see that a basis of entourages for $\mathcal{C}^*\mathcal{U}$ consists entourages $\mathcal{E}_{\mathcal{A}} = \{(x,y) \mid x, y \in \mathcal{E}[A], A \in \mathcal{A}\}$ so that $\mathcal{A}$ is any finite cover of X and $\mathcal{E}$ is any entourage in $\mathcal{U}$. But then, if $\mathcal{E}'$ is an entourage of the uniform space $(\mathcal{S}(X), \overline{\mathcal{C}^*\mathcal{U}})$ the entourage $\mathcal{E}' \restriction X \times X$ can be refined by an entourage of the form $\mathcal{E}_{\mathcal{A}}$, for some $\mathcal{E} \in \mathcal{U}$ and some finite cover $\mathcal{A}$ of X. The rest follows from (6).

(7) ⇒ (8): it follows from the universal property of $\mathcal{S}(X)$: for every $f\colon X \to Y$ with Y compact there exists $\tilde{f}\colon \mathcal{S}(X) \to Y$ with $f = \tilde{f} \circ i$.

(8) ⇒ (1): if $f\colon X \to \mathbb{R}$ is bounded then range(f) is compact. □

We can now connect finite oscillation stability, and therefore the Ramsey property with extreme amenability of a Polish group G by focusing on the uniform G-space $(G, \mathcal{U}_L)$ where G is acting on G by left translation $(g, x) \mapsto gx$, and $\mathcal{U}_L$ is the **left uniform structure** on G, i.e., the uniform structure generated by entourages of the form

$$V_L := \{(x, y) \in G \times G \mid x^{-1}y \in V\},$$

where V ranges over all open neighborhoods of the identity of G. Notice that V_L is left invariant, i.e., $(gx, gy) \in V_L$ for all $g, x, y \in G$. The **right uniform structure** $(G, \mathcal{U}_R)$ on G is defined analogously by considering the right invariant entourages

$$V_R := \{(x, y) \in G \times G \mid xy^{-1} \in V\}.$$

Let $\mathcal{S}_R(G)$ be the **right Samuel compactification** of the uniform space $(G, \mathcal{U}_R)$.

If ρ is any left invariant bounded pseudo-metric on G and $K_\rho = \{x \in G \mid \rho(x, 1) = 0\}$, then ρ induces a metric $\hat{\rho}$ on the left coset space G/K_ρ given by $\hat{\rho}(xK_\rho, yK_\rho) = \rho(x, y)$. Notice that G acts from the left of $(G/K_\rho, \hat{\rho})$ by isometries.

Theorem 50. *Let G be a Polish group. The following are equivalent.*

1. *G is extremely amenable.*
2. *The action of G on $(G, \mathcal{U}_L)$ by left translation is finitely oscillation stable.*
3. *Any continuous and transitive action of G on a metric space (X, d) by isometries is finitely oscillation stable.*
4. *There is a directed collection $\mathcal{R}$ of bounded left-invariant continuous pseudometrics on G which determine the topology on G so that for every $\rho \in \mathcal{R}$ the action of G on $(G/K_\rho, \hat{\rho})$ is finitely oscillation stable.*

Proof. (1) ⇒ (2): first notice that the fixed left action $(g, x) \mapsto gx$ of G on G is also right uniformly continuous, i.e., $(G, \mathcal{U}_R)$ is a uniform G-space as well. To see this let $g \in G$ and let V be any open neighborhood of the identity. Let V' be any open neighborhood of the identity with $gV'g^{-1} \subseteq V$ and notice that

$$(x, y) \in V'_R \implies xy^{-1} \in V' \implies gxy^{-1}g^{-1} \in gV'g^{-1} \implies (gx, gy) \in V_R.$$

As a consequence this action extends to an action of G on $\mathcal{S}_R(G)$. Since G is assumed to be extremely amenable we get a fixed point $\xi \in \mathcal{S}_R(G)$. We fix (g_a) to be a net in G converging to ξ.

Let $f\colon G \to \mathbb{R}$ be a bounded map that is uniformly continuous with respect to $\mathcal{U}_L$ on G, let F be a finite subset of G and let $\varepsilon > 0$. Since $x \mapsto x^{-1}$ is a isomorphism between the uniform spaces $(G,\mathcal{U}_L)$ and $(G,\mathcal{U}_R)$ we have that the map f^* defined by $f^*(x) = f(x^{-1})$ is a uniformly continuous with respect to $(G,\mathcal{U}_R)$ and therefore it extends to a continuous map $f^*\colon \mathcal{S}_R(G) \to \mathbb{R}$. By continuity of the action $G \curvearrowright \mathcal{S}_R(G)$ we have for every $x \in G$ that $xg_a \to x\xi = \xi$ and since F is finite we can pick g_a so that for all $x \in F^{-1}$ we have that $|f^*(xg_a) - f^*(\xi)| < \varepsilon/2$. Hence, for $g = g_a^{-1}$ and for every $x, y \in F$ we have that $|f(gx) - f(gy)| < \varepsilon$ as required.

(2) $\Rightarrow$ (3): let $f\colon X \to \mathbb{R}$ be any bounded uniformly continuous map and let x be any point of X. Notice that the map $f_x\colon G \to \mathbb{R}$ with $f_x(g) = f(gx)$ is also bounded and left uniformly continuous since the map $g \mapsto gx$ is left uniformly continuous: by continuity of the action, for every ε there is a V so that if $g^{-1}h \in V$ then $d(g^{-1}hx, x) < \varepsilon$; but since the action is by isometries, this implies that $d(hx, gx) < \varepsilon$. Since the action is transitive, for every finite $F \subseteq X$ there is a finite $F' \subseteq G$ so that $F = F'x$. As a consequence finite oscillation stability on $(G,\mathcal{U}_L)$ pushes forward to (X, d).

(3) $\Rightarrow$ (4): let $\mathcal{R}$ be any directed collection of bounded left invariant uniformly continuous pseudo-metrics that generate the topology on G and recall that for every $\rho \in \mathcal{R}$ the group G acts from the left of $(G/K_\rho, \hat{\rho})$ by isometries.

(4) $\Rightarrow$ (2) : This follows from (2) $\implies$ (1) of Theorem 49.

(2) $\Rightarrow$ (1) : Assume that X is a compact G-space let x be any point of X. Notice that the map $g \mapsto g^{-1}x$ is left uniformly continuous. To see this let $\mathcal{E}$ be an element of the unique uniformity compatible with the topology on X (since X is compact) and notice that by the continuity of the action, for all $z \in X$ there is an open neighborhood V_z of the identity of G and U_z an open neighborhood of z so that $W_z \cdot U_z \subseteq \mathcal{E}[\{x\}]$. By compactness we pass to a subcover, finding a single V so that for all $z \in X$ we have that $(z, gz) \in \mathcal{E}$. But now, going back to the map $g \mapsto g^{-1}x$, if $g, h \in G$ and $g^{-1}h \in V$ then we have that $(g^{-1}x, h^{-1}x) = (g^{-1}h(h^{-1}x), (h^{-1}x)) = (g^{-1}hy, y) \in \mathcal{E}$, where $y = h^{-1}x$.

By the assumption and (8) of Theorem 49 we have that $g \mapsto g^{-1}x$ is finitely oscillation stable. Hence, for every entourage $\mathcal{E}$ in X and every finite $F \subseteq G$ there is a $g_{\mathcal{E},F} \in G$ so that $F^{-1}g^{-1}_{\mathcal{E},F}x$ is $\mathcal{E}$-small. Since X is compact we can pick a cluster point ξ of the net $g^{-1}_{\mathcal{E},F}x$, i.e. a ξ so that for all open $U \subseteq X$ containing ξ and every $\mathcal{E}, F$ as above there exist $\mathcal{E}' \subseteq \mathcal{E}$, $F' \supseteq F$ so that $g^{-1}_{\mathcal{E}',F'}x \in U$.

But then it is easy to see that ξ is a fixed point. For that it suffice to show that for every $g \in G$ and every open neighborhood U of ξ we have that $U \cap gU \neq \emptyset$. Let now $\mathcal{E}, F$ as above, with $F \supseteq \{g^{-1}, 1\}$ and $\mathcal{E}[\mathcal{E}[\{\xi\}]] \subseteq U$. Since ξ is a cluster point of the above net, we get $\mathcal{E}' \subseteq \mathcal{E}$, $F' \supseteq F$ so that $g^{-1}_{\mathcal{E}',F'}x \in \mathcal{E}[\xi]$. But then, by the choice of $g_{\mathcal{E}',F'}$ we have that $(F')^{-1}g^{-1}_{\mathcal{E},F}x$ is $\mathcal{E}$-small and since F' contains 1 and g^{-1} we have that $gg^{-1}_{\mathcal{E}',F'}x \in \mathcal{E}[g^{-1}_{\mathcal{E}',F'}x]$. Since $g^{-1}_{\mathcal{E}',F'}x \in \mathcal{E}[\xi]$ we have that $g\mathcal{E}[\xi] \cap \mathcal{E}[\mathcal{E}[\xi]] \neq \emptyset$ as required. $\square$

7. Week 7

We can now finish the proof of Theorem 42. Notice that for every Polish group G, if V is an open subgroup of G then we can define a left invariant pseudo-metric ρ_V on G with $\rho_V(g,h) = 0$ if $gV = hV$ and 1 otherwise. The first observation now is that when G is a closed subgroup of S_∞ , the collection $\mathcal{R}_G$ containing all ρ_V as above generates the topology on G. In fact we can assume (and we do so) that $\mathcal{R}_G$ contains all pseudo-metrics ρ_V where V is ranges over all stabilizers of finite subsets $A \subseteq M$. Moreover, is $G = \mathrm{Aut}(\boldsymbol{M})$ where $\boldsymbol{M}$ is the Fraïssé limit of the Fraïssé class $\mathcal{K}$, if V corresponds to the stabilizer the finite set $A \subseteq M$ and $\boldsymbol{A}$ is the substructure on domain $\boldsymbol{A}$ then the space $(G/K_\rho, \hat{\rho})$ canonically corresrponds to the set

$$\mathrm{Emb}(\boldsymbol{A}, \boldsymbol{M}) = \{f \colon \boldsymbol{A} \to \boldsymbol{M} \mid \ f \text{ is an embedding}\},$$

endowed with the discrete metric.

Proof of Theorem 42. (1) $\Rightarrow$ (2) : if $\mathrm{Aut}(\boldsymbol{M})$ is extremely amenable and since for every $\boldsymbol{A} \in \mathcal{K}$ the group $\mathrm{Aut}(\boldsymbol{M})$ acts on the metric space $\mathrm{Emb}(\boldsymbol{A}, \boldsymbol{M})$ transitively, by Theorem 50 (3) we have that $\mathrm{Emb}(\boldsymbol{A}, \boldsymbol{M})$ is finitely oscillation stable. If $\boldsymbol{B} \in \mathcal{K}$ with $\boldsymbol{A} \leq \boldsymbol{B}$, fix an embedding $i \colon \boldsymbol{B} \to \boldsymbol{M}$ and let F be the finite subset of $\mathrm{Emb}(\boldsymbol{A}, \boldsymbol{M})$ consisting of all embeddings of $\boldsymbol{A}$ in $\boldsymbol{M}$ which factor through i. Given any number of colors l, by the characterization Theorem 49 (8) of finite oscillation stability we can find a finite $K \subset \mathrm{Emb}(\boldsymbol{A}, \boldsymbol{M})$ so that whenever K is colored in l-many colors there is a $g \in \mathrm{Aut}(\boldsymbol{M})$ so that gF is a subset of K (since we are working with a discrete space) and monochromatic. By enlarging K if necessary, we can assume that it consists of all embeddings of $\boldsymbol{A}$ in $\boldsymbol{M}$ which factor through some fixed embedding i' of some $\boldsymbol{C} \in \mathcal{K}$ in $\boldsymbol{M}$. Then $\boldsymbol{C}$ is the required structure witnessing the Ramsey property for the triple $(\boldsymbol{A}, \boldsymbol{B}, l)$.

(2) $\Rightarrow$ (1) : conversely if $\mathcal{K}$ satisfies the Ramsey property then the identification

$$\mathrm{Emb}(\boldsymbol{A}, \boldsymbol{M}) \Leftrightarrow (G/K_\rho, \hat{\rho})$$

established above provides us with a collection $\mathcal{R}_G$ of bounded left-invariant continuous pseudo-metrics on G which determine the topology on $\mathrm{Aut}(\boldsymbol{M})$ so that for every $\rho \in \mathcal{R}_G$ the action of $\mathrm{Aut}(\boldsymbol{M})$ on $(G/K_\rho, \hat{\rho})$ is finitely oscillation stable. By by Theorem 50 (4) we have that $\mathrm{Aut}(\boldsymbol{M})$ is extremely amenable. □

Another example of a Fraïssé class which has the Ramsey property is the collection of all linearly ordered finite graphs or more generally linearly n-hypergraphs. A simple proof of this using only the pigeonhole principle can be found in the paper "A Short Proof of the Restricted Ramsey Theorem for Finite Set Systems" of Prömel and Voigt. The Fraïssé class of all finite structures in the empty language does not have the Ramsey property since if $\boldsymbol{A} = \boldsymbol{B}$ is the structure consisting of two points and $\phi \in \mathrm{Aut}(\boldsymbol{B})$ is the unique non-trivial automorphism of $\boldsymbol{A}$ then for any $\boldsymbol{C}$ with $\boldsymbol{A} \leq \boldsymbol{C}$ one can find a bad two coloring of $\mathrm{Emb}(\boldsymbol{A}, \boldsymbol{C})$ by coloring different color every two embeddings $f, f' \in \mathrm{Emb}(\boldsymbol{A}, \boldsymbol{C})$ with $f = f' \circ \phi$. Similarly by a theorem of Veech there is no locally compact group that is extremely amenable and therefore the age $\mathcal{K}$ of any Fraïssé limit which has locally compact automorphism group cannot have the Ramsey property.

For future purposes and since we have the notation set we finish by proving the following theorem regarding Banach space representations of Polish groups.

Theorem 51. *Let G be a Polish group then the map $\lambda\colon G \to \mathrm{Iso}_{\mathrm{linear}}(\mathrm{RUC}_b(G))$*

$$\lambda(g)(\phi) = {}^g\phi, \quad \textit{with} \quad {}^g\phi(x) = \phi(gx)$$

Is a strongly continuous and faithful representation of G on the Banach space $\mathrm{RUC}_b(G)$ of all bounded uniformly continuous functions on G endowed with the supremum norm.

Proof. Let $\phi \in \mathrm{RUC}_b(G)$. Right uniform continuity implies that for every $\varepsilon > 0$ there exists an open neighborhood V of 1 so that whenever $g, h \in G$ with $gh^{-1} \in V$ then $|\phi(g) - \phi(h)| \leq \varepsilon$. But then to see that λ is strongly continuous notice that for ε, V, g, h as above, and for every $x \in G$ we have that $|{}^g\phi(x) - {}^g\phi(x)| = |\phi(gx) - \phi(hx)| = \varepsilon$ since $gx(hx)^{-1} = gh^{-1}$. □

Notice that in the above theorem it is important that we act from the *left* to the space of *right* uniformly continuous functions.

A big part of mathematical activity revolves around classification problems. However, not every classification problem has a satisfactory solution and some classification problems are provably more complicated than others. The following formal setup encompasses many natural classification problems and allows us to treat these complexity considerations mathematically.

Definition 52. A **classification problem** is a pair (X, E) where X is a Polish space and E is an analytic equivalence relation. Recall that an equivalence relation E is analytic if it is the continuous image of some Polish space under a continuous map.

Example 53. We list here several natural classification problems which fit within the above context.

(1) $(\mathrm{Graphs}(\omega), \simeq_{\mathrm{iso}})$ denotes the classification problem of **isomorphism of countable graphs**. The space $\mathrm{Graphs}(\omega)$ consists of all symmetric and irreflexive binary relations on ω and therefore it a closed subspace of the Polish space $2^{\omega\times\omega}$. It is easy to see that $\simeq_{\mathrm{iso}}$ is an analytic equivalence relation on $\mathrm{Graphs}(\omega)$.

(2) $(\mathrm{Str}_{\mathcal{L}}(\omega), \simeq_{\mathrm{iso}})$ denotes more generally the classification problem of **isomorphism between countable $\mathcal{L}$-structures**. Here we assume that $\mathcal{L}$ is countable (and relational) hence $\mathrm{Str}_{\mathcal{L}}(\omega)$ can be identified with the product

$$\Pi_{R\in\mathcal{L}} 2^{\alpha(R)},$$

where α stands for the arity of R.

(3) $(\mathcal{U}(\mathcal{H}), \simeq_U)$ denotes the classification problem of **unitary equivalence between unitary operators**. If $\mathcal{H}$ is a separable Hilbert space then $\mathcal{U}(\mathcal{H})$ is a Polish space with respect to the strong operator topology (pointwise convergence topology) and the equivalence relation $\simeq_U$ defined by

$$T \simeq_U S \iff \exists U \in \mathcal{U}(\mathcal{H}) \quad UTU^{-1} = S,$$

is an analytic (in fact Borel) equivalence relation.

(4) $(\mathrm{C}^\infty\mathrm{Str}(\mathrm{R}^4), \simeq_{\mathrm{diffeo}})$ denotes the diffeomorphism problem between different smooth structures of $\mathbb{R}^4$.

In order to *solve* the problem (X, E), i.e., in order to classify the elements of X up to E, one needs to find a complete set of definable invariants for X. The assignments $G \mapsto \mathrm{maxdeg}(G)$ and $G \mapsto \mathrm{conn}(G)$ which map each graph $G \in \mathrm{Graphs}(\omega)$ to its maximum degree and to its number of connected components, are both definable invariants. However, the collection $\{\mathrm{maxdeg}(\cdot), \mathrm{conn}(\cdot)\}$ is not a *complete* set of invariants since there are non-isomorphic graphs with the same number of connected components and the same maximum degree. In fact, as we are going to see, there is no "satisfactory" collection of complete invariants for $(\mathrm{Graphs}(\omega), \simeq_{\mathrm{iso}})$, or in terms of the following definition, $(\mathrm{Graphs}(\omega), \simeq_{\mathrm{iso}})$ is not *concretely classifiable*; [FS89].

Definition 54. A classification problem (X, E) is **concretely classifiable** if there is a Borel map $f\colon X \to 2^\omega$ so that $xEx' \iff f(x) = f(x')$.

Notice that the above definition is very flexible in terms of what invariants can be used for concrete classification, since every uncountable Polish space is Borel isomorphic to 2^ω, and since the class of Polish spaces is closed under countable products.

Examples of concretely classifiable classification problems are plenty. For example when the Hilbert space $\mathcal{H}$ is of finite dimension $n > 0$ then the map $T \mapsto (\lambda_1, \ldots, \lambda_n)$ which maps each element of $\mathcal{U}(\mathcal{H})$ to its eigenvalues $\bar{\lambda} \in \mathbb{T}^n$ in (clockwise) increasing order is a concretely classifies $(\mathcal{U}(\mathcal{H}), \simeq_U)$. However, as we will see $(\mathcal{U}(\mathcal{H}), \simeq_U)$ is not concretely classifiable when $\mathcal{H}$ is infinite dimensional. One would still hope that $(\mathcal{U}(\mathcal{H}), \simeq_U)$ could be classified *relative to* an isomorphism problem between countable structures such as $(\mathrm{Graphs}(\omega), \simeq_{\mathrm{iso}})$. However, as we will prove there is no definable way to do so, i.e., in terms of the following definition there is no *Borel reduction* from $(\mathcal{U}(\mathcal{H}), \simeq_U)$ to $(\mathrm{Str}_{\mathcal{L}}(\omega), \simeq_{\mathrm{iso}})$ for any $\mathcal{L}$.

Definition 55. Let (X, E) and (Y, F) be two classification problems. A **reduction** from E to F is any map $f\colon X \to Y$ with $xEx' \iff f(x)Ff(x')$. We say that it is a **Borel reduction** if f is moreover Borel, as a map from X to Y. We write $E \leq_B F$ when such a Borel reduction exists.

The relation $\leq_B$ induces a preordering on the class of classification problems, reflecting their relative complexity.

Particularly interesting is the case when the equivalence relation E on X is obtained as the *orbit equivalence relation* E^G_X of a a Polish G-space X, where

$$xE^G_X y \iff [x]_G = [y]_G \iff \exists g \in G \;\; gx = y.$$

In fact the examples mentioned so far are of this form. For example, the problem $(\mathcal{U}(\mathcal{H}), \simeq_U)$ is induced by the action of the unitary group $\mathcal{U}(\mathcal{H})$ on $\mathcal{U}(\mathcal{H})$ by conjugation, the problem $(\mathrm{Str}_{\mathcal{L}}(\omega), \simeq_{\mathrm{iso}})$ is induced by the *logic action* of the group of permutations S_∞ of ω on the space $\mathrm{Str}_{\mathcal{L}}(\omega)$, and $(2^\omega, =)$ is induced by the trivial action of the trivial group on 2^ω. This observation provides a link between topological dynamics and obstruction theory. In short, some obstructions to classification are reflections of the restrictions that the topology of G puts on a G-space X, and therefore, on E^G_X. For example, in the context of orbit equivalence relations, a classical dynamical obstruction to concrete classification is a combination of *generic ergodicity* with lack of comeager orbits. An orbit equivalence relation E^G_X is **generically ergodic** if it satisfies any of the following equivalent conditions.

Lemma 56. *Let X be a Polish G-space. Then the following are equivalent:*

(1) *every Baire-measurable G-invariant subset of X is either meager or comeager;*
(2) *every invariant non-empty open set is dense;*
(3) *there are comeager many $x \in X$ so that $[x]_G$ is dense;*

(4) there is a dense orbit;
(5) for every non-empty open $U, U' \subseteq X$ there is $g \in G$ with $gU \cap U' \neq \emptyset$.

Proof. (1) $\Rightarrow$ (2) : if U is open non-empty then it is non-meager. Hence if it is moreover invariant, by (1) it has to be comeager.

(2) $\Rightarrow$ (3) : if U is a non-empty open set then $[U]_G$ is also non-empty and open but also invariant. Hence by assumption comeager. Moreover if $x \in [U]_G$ then $[x] \cap U \neq \emptyset$. Let (U_n) be a countable basis for the topology on X consisting of non-empty open sets and let $C = \cap_n [U_n]$. Then C is G_δ and for every $x \in C$ we have that $x \in [U_n]$, so $[x] \cap U_n \neq \emptyset$. Therefore for every $x \in C$ we have that $[x]$ is dense.

(3) $\Rightarrow$ (4) : clear.

(4) $\Rightarrow$ (5) : if $[x]$ is dense then $gx \in U$ and $g'x \in U'$ for some $g, g' \in G$. Hence $g'g^{-1}U \cap U'$ contains $g'x$.

(5) $\Rightarrow$ (1) : assume that there is an invariant Baire measurable set A so that both A and $B = A^c$ are non meager. By Baire-measurability we can find U_A and U_B so that A is comeager in U_A and B is comeager in U_B. Let $g \in G$ with $gU_A \cap U_B \neq \emptyset$ and let U be this non-empty and open common intersection. Since gA is comeager in gU_A it is comeager in U as well. But $A = gA$ so A is comeager in U which contradicts the assumption that its complement B is comeager in U. □

Theorem 57. *If X is a generically ergodic Polish G-space with meager orbits then E^G_X is not concretely classifiable.*

Proof. Assume that $f\colon X \to 2^\omega$ be a Borel reduction. For every $s \in 2^{<\omega}$ let N_S be the basic open set of 2^ω consisting of all sequences extending s.

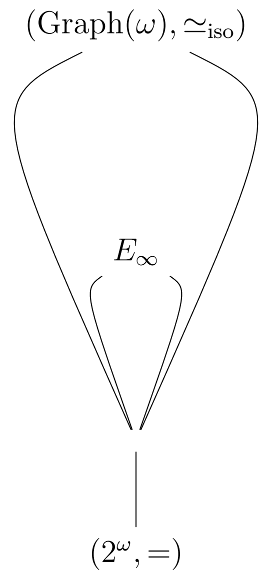
Notice that $\{f^{-1}(N_{(0)}), f^{-1}(N_{(1)})\}$ forms a partition of invariant Borel subsets of X. By Lemma 56 one of them has to be comeager. Continuing inductively we build a sequence $\alpha \in 2^\omega$ so that for all $n > 0$ we have that $f^{-1}(N_{\alpha|n})$ is comeager. But then $C = \cap_n f^{-1}(N_{\alpha|n})$ is a comeager subset of X that is mapped under f to the singleton $\{\alpha\}$. Since the orbits in X are meager C contains x, y which are not E^G_X-equivalent, yet they map to the same α showing that f is not a reduction. □

Corollary 58. *(1) The orbit equivalence relation of the shift action $\mathbb{Z} \curvearrowright 2^{\mathbb{Z}}$ is not concretely classifiable.*
(2) $(\mathrm{Graphs}(\omega), \simeq_{\mathrm{iso}})$ *is not concretely classifiable.*
(3) $(\mathcal{U}(\mathcal{H}), \simeq_U)$ *is not concretely classifiable if $\mathcal{H}$ is infinite dimensional.*

Proof. We leave the proof as an exercise. For (1) simply use (5) of 56 to show generic ergodicity and then invoke Theorem 57. For part (3) it takes some arguing to show that orbits are meager but generic ergodicity follows easily by (5) of 56. For (2) try to Borel reduce (1) to it. □

Corollary 58 provides examples of classification problems which are not concretely classifiable. Given such a problem (X, E) the pertinent question then becomes whether there is some other simple classification problem (Y, F) which we can use as invariants to classify (X, E), i.e., given (X, E) what is the simplest (Y, F) so that $(X, E) \leq_B (Y, F)$?

If (X, E) is not concretely classifiable then the next hope often is that (X, E) is **essentially countable**, that is, $(X, E) \leq_B (Y, F)$ with F being "countable." Recall that an equivalence relation F on Y is said to be **countable** if the equivalence class $[y]_F$ of every y in Y is a countable set. The collection of essentially countable classification problems admits a $\leq_B$-maximum element usually denoted by E_∞. Next comes the collection of all classification problems which are **classifiable by countable structures**, i.e., (X, E) for which there is some countable language $\mathcal{L}$ so that $(X, E) \leq_B (\mathrm{Str}_{\mathcal{L}}(\omega), \simeq_{\mathrm{iso}})$. The problem $(\mathrm{Graph}(\omega), \simeq_{\mathrm{iso}})$ happens to be the $\leq_B$-maximum among all problems which are classifiable by countable structures. However many natural classification problems are intrinsically more complicated than the problems in these lower levels of the $\leq_B$-complexity hierarchy.



Theorem 59 (Kechris-Sofronidis)**.** *When $\mathcal{H}$ is infinite dimensional then $(\mathcal{U}(\mathcal{H}), \simeq_U)$ is not classifiable by countable structures.*

The proof of this theorem uses Hjorth's theory of *turbulence*. Turbulence is a certain strengthening of generic ergodicity which constitutes a dynamical obstruction for classifiability by countable structures, and more generally, by orbit equivalence relations of non-Archimedean Polish groups. To put things into the right dynamical framework consider the following definition.

Definition 60. Let $\mathcal{C}$ be a class of Polish groups and let (X, E) be a classification problem. We say that (X, E) is **classifiable by $\mathcal{C}$-group actions** if there is a group H in $\mathcal{C}$ and a Polish H-space Y so that E Borel reduces to an orbit equivalence relation E^H_Y.

This definition is the starting point of a whole project in the Borel complexity theory of classification problems.

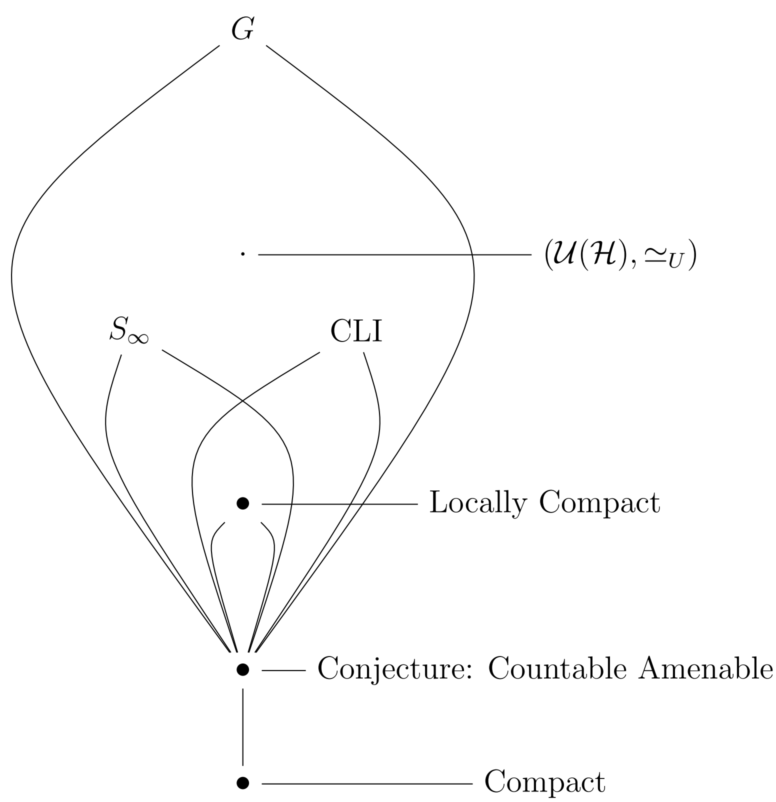
Problem 61. Given a class $\mathcal{C}$ of Polish groups, which dynamical conditions on a Polish G-space X ensure that the orbit equivalence relation E^G_X is not classifiable by $\mathcal{C}$-group actions?

Indeed, generic ergodicity provides such a condition for the class $\mathcal{C}$ of *compact Polish groups* . To see this notice that if G is a compact Polish group and X is a

Polish G-space, then the associated orbit equivalence relation is always concretely classifiable. This follows from two facts. Namely, that the space $\mathcal{K}(X)$ of all compact subsets of X equipped with the Vietoris topology is a Polish space [Kec11] and that the assignment $x \mapsto [x]_G$ from X to $\mathcal{K}(X)$ is Borel. Hjorth's turbulence theory addresses this same problem in the case when $\mathcal{C}$ is the class of Polish groups which, like S_∞, are *non-Archimedean*. A similar dynamical obstructions is provided in [LP] for the case when $\mathcal{C}$ consists of all CLI-groups. In fact similarly to Theorem 59 we have the following result.

Theorem 62 ([LP])**.** *When $\mathcal{H}$ is infinite dimensional then $(\mathcal{U}(\mathcal{H}), \simeq_U)$ is not classifiable by CLI-group actions.*

We can now redraw the various complexity classes of orbit equivalence relations from the point of view of Definition 60. Concretely classifiable problems are precisely the ones classifiable by compact group actions, essentially countable problems are the ones classifiable by locally compact groups [Kec92], and finally the problems which are classifiable by countable structures correspond to the ones classifiable by S_∞-actions.



We can now state the aforementioned obstructions for classifiability under S_∞-actions and for classifiability under CLI-actions.

Let X be a Polish G-space and let $x \in X$, U an open neighborhood of x in X, and V an open neighborhood of 1 in G. The (U, V)**-local orbit** of x is defined to be the set

$$\mathcal{O}(x, U, V) = \{y \in U \mid \exists n > 0 \;\; \exists g_1, g_2, \ldots, g_n \in V$$
$$(g_n \cdots g_1 x = y) \wedge (\forall i < n \; g_i \cdots g_1 x \in U)\big\}.$$

Theorem 63 (Hjorth's turbulence theorem)**.** *Let X be a Polish G-space which is* **turbulent***. That is:*

(1) there is a dense orbit in X;
(2) all orbits are meager in X;
(3) for a generic $x \in X$, every U open neighborhood of x in X, and every open neighborhood V of 1 in G, we have that $\mathcal{O}(x, U, V)$ is somewhere dense, i.e.,

$$\mathrm{int}(\overline{\mathcal{O}(x, U, V)}) \neq \emptyset.$$

Then E_X^G is not classifiable by non-Archimedean group actions.

Notice that for non-Archimedean groups we can approximate the identity by open subgroups V for which the notion of the local orbit trivializes, i.e., $\mathcal{O}(x, U, V) = Vx \cap U$. This observation together with the next lemma indicates explains why non-Archimedean group action are never turbulent. Of course we will need to work harder to show that one cannot reduce on them turbulent actions.

Lemma 64. *Let X be a Polish G-space and let $x \in X$. Then the following are equivalent:*

(1) The orbit $[x]$ is non-meager;
(2) For each open neighborhood V of 1, we have that Vx is non-meager;
(3) For each open neighborhood V of 1, we have that Vx is somewhere dense.

Proof. (1) $\implies$ (2) : let (g_n) be a sequence in G so that $\bigcup_n g_n V = G$. Then (1) implies that there is n so that $g_n V x$ in non-meager. Hence, since g_n acts with homeomorphisms we have Vx is non-meager.

(2) $\implies$ (3) : clear.

(3) $\implies$ (1) : Assume towards contradiction that (F_n) is a sequence of closed nowhere dense subsets of X with $[x] \subseteq \bigcup_n F_N$. By continuity of $g \mapsto gx$ we have that $Q_n = \{g \in G \mid gx \in F_n\}$ is closed. Since $\bigcup_n Q_n = G = G$, by Baire category theorem there is n so that $W = \mathrm{int}(Q_n) \neq \emptyset$. Let $V = g^{-1}W$ for any $g \in W$. By assumption (3) we have that Vx is somewhere dense in X and therefore $Wx = gVx$ is dense in some non-empty open $U \subseteq X$. Which implies that F_n is dense in U since $Wx \subseteq F_n$. □

To formulate now the dynamical obstruction for classifiability by CLI-group actions consider the following definition. If x, y are points in the G-space X, then we say that x **left-embeds in** y if there is a left-Cauchy sequence (g_n) in G so that $g_n x \to y$. Similarly we say that x **right-embeds in** y if there is a right-Cauchy sequence (g_n) in G so that $g_n y \to x$ (notice the change of the roles of x and y). In the rest it will be convenient to work with right-embeddings.

Theorem 65. *Let X be a Polish G-space which is* **generically 1-dimensional***, i.e., for every comeager subset C of X there is $x, y \in X$ so that:*

(1) $[x] \neq [y]$;

(2) x right-embeds in y.

Then E^G_X is not classifiable by CLI group actions.

Here it should be clear that that CLI groups cannot be generically 1-dimensional (they are actually 0-dimensional).

8. Week 8

The proof for both Theorem 65 and Theorem 63 uses the next lemma. Recall that if $(X,E),(Y,F)$ are classification problems then an (E,F)**-homomorphism** is just a map $f\colon X\to Y$ with $xEy \implies f(x)F(y)$.

Recall that for any property P of elements of a Polish space X, the notation $\forall^* x\in X P(x)$ stands for "the set $\{x\in X \mid P(x)\}$ is comeager in X"

Lemma 66. *Suppose that G,H are Polish groups, X is a Polish G-space, and Y is a Polish H-space. Let $f: X\to Y$ be a Baire-measurable $\big(E^X_G, E^Y_H\big)$-homomorphism. Then there exists a dense G_δ subset C of X such that*

(1) the restriction of f to C is continuous;
(2) for any $x\in C$, $\{g\in G : gx\in C\}$ is a comeager subset of G;
(3) for any open neighborhood W of the identity in H there an open cover $\mathcal{U}$ of C and a collection $\{V_U \mid U\in\mathcal{U}\}$ of open neighborhoods of the identity of G so that for any $U\in\mathcal{U}$, any $x\in U$, and for a comeager set of $g\in V$:
  (i) $f(gx)\in Wf(x)$;
  (ii) $gx\in C$.

For the proof of Lemma 66 we will use a couple of classical "Baire-category facts".

Lemma 67. *If $f\colon X\to Y$ is a Baire-measurable map between Polish spaces then there exists a dense G_δ subset C of X so that $f\restriction C$ is continuous.*

Proof. Let (U_n) be a countable basis of open sets in Y. Since $B_n = f^{-1}(U_n)$ is Baire-measurable, there exists a meager F_σ subset F_n of X and an open subset V_n of X so that $B_n\setminus F_n = V_n\setminus F_n$. Let $C = X\setminus\bigcup_n F_n$ and notice that $(f\restriction C)^{-1}U_n$ is open in C for all n. □

Lemma 68. *If $f\colon X\to Y$ is continuous and open map between Polish spaces and $C\subseteq X$ is Baire-measurable then the following are equivalent:*

(1) C is comeager in X;
(2) $\forall^* y\in Y\ \big(f^{-1}(y)\cap C$ is comeager in $f^{-1}(y)\big)$.

Proof. (1) $\implies$ (2): by Baire category theorem it suffice to prove (2) for the special case when C is dense and open in X. Then, for all $y\in Y$, $f^{-1}(y)\cap C$ is automatically open in $f^{-1}(y)$ so we are left to show that $\forall^* y\in Y\ \big(f^{-1}(y)\cap C$ is dense in $f^{-1}(y)\big)$. If $\mathcal{U}$ is a countable basis of open sets in X then the last statement is equivalent to

$$\forall U\in\mathcal{U}\ \forall^* y\in Y\big(f^{-1}(y)\cap U\neq\emptyset \implies (f^{-1}(y)\cap C)\cap U\neq\emptyset\big).$$

But since the map f is open, for every fixed $U\in\mathcal{U}$ we have that the set

$$C^Y_U = \{y\in Y \mid \big(f^{-1}(y)\cap U\neq\emptyset \implies (f^{-1}(y)\cap C)\cap U\neq\emptyset\big)\}$$

is G_δ (being the union of an open and a closed set). So we are left to show that C^Y_U is dense. Let $V\subseteq Y$ open. If $f^{-1}(V)\cap U=\emptyset$ then for all $y\in V$ we have $f^{-1}(y)\cap U=\emptyset$ and therefore V is in fact a subset of C^Y_U. In particular $C^Y_U\cap V\neq\emptyset$.

On the other hand, since $f^{-1}(V) \cap U$ is an open set, if it is non-empty then C intersects it. If $x \in \big(f^{-1}(V) \cap U\big) \cap C$ then $f(x)$ is an element of C^Y_U lying in V.

(2) $\implies$ (1): if C is not comeager in X then since C is Baire-measurable we can find a non-empty open subset U of X so that $C \cap U$ is meager in U. We can apply now (1) $\implies$ (2) to (the comeager complement of) $C \cap U$ to get that

$$\forall^* y \in Y \; \big(f^{-1}(y) \cap (C \cap U) \text{ is meager in } f^{-1}(y)\big).$$

But then, since $f(U)$ is open, the last condition and the condition (2) from the assumption implies that there exists some $y \in f(U)$ so that $(f^{-1}(y) \cap C) \cap U$ is meager in $f^{-1}(y)$ and at the same time $f^{-1}(y) \cap C$ is comeager in $f^{-1}(y)$. Since $U \cap f^{-1}(y) \neq \emptyset$ this contradicts the Baire-category theorem in the Polish space $f^{-1}(y)$. □

If X is a Polish G-space and $A \subseteq X$ then the Vaught transform $A^{\forall^*}$ of A is the set $\{x \in X \mid \forall^* g \in G \; gx \in A\}$.

Lemma 69. *If X is a Polish G-space and $A \subseteq X$ then $A^{\forall^*}$ is an invariant subset of X. Moreover, if A is G_δ then so is $A^{\forall^*}$.*

Proof. Let $x \in A^{\forall^*}$ and let $g \in G$. If $O = \{h \in G \mid hx \in A\}$ is comeager subset of G then Og^{-1} is also comeager in G and $Og^{-1}(gx) \subseteq A$. Hence $gx \in A^{\forall^*}$.

Let now $A = \cap_n U_n$, where U_n are open subsets of X. If $\mathcal{V}$ is a countable basis for the topology of G and $\mathcal{U}$ is a countable basis for the topology of X, then we have:

$$\begin{aligned}
\{x \in X \mid \forall^* g \in G \; gx \in A\} &= \\
\{x \in X \mid \forall^* g \in G \; \forall n \in \mathbb{N} \; gx \in U_n\} &= \\
\{x \in X \mid \forall n \in \mathbb{N} \; \forall^* g \in G \; gx \in U_n\} &= \\
\{x \in X \mid \forall n \in \mathbb{N} \; \forall V \in \mathcal{V} \; \exists^* g \in V \; gx \in U_n\} &= \\
\{x \in X \mid \forall n \in \mathbb{N} \; \forall V \in \mathcal{V} \; \exists V' \in \mathcal{V} \; \exists U \in \mathcal{U} \; (V'U \subseteq U_n \text{ and } x \in U)\}&,
\end{aligned}$$

where the last equality follows from the continuity of the action and it is clearly a G_δ condition. □

We can now proceed with the proof of Lemma 66.

Proof of Lemma 66. Let us focus on property (3) since by Lemma 67 we can always restrict to a dense G_δ subspace of X satisfying property (1). Moreover, if C_0 is any dense G_δ subset of X satisfying properties (1) and (3), the Vaught transform $C_0^{\forall^*}$ of C_0 is also dense G_δ by Lemma 69 and Lemma 68 (notice that the action is continuous and open as a map $G \times X \to X$). Hence, letting $C = C_0 \cap C_0^{\forall^*}$ we get the required set.

(3) is a "locally uniform" property. We first establish the weaker "orbitwise" analogue.

Fix an open neighborhood W of identity of H.

Claim. *For comeager many $x \in X$ there exists an open neighborhood V of the identity of G so that $\forall^* g \in V$ we have that $f(gx) \in Wf(x)$.*

Proof of Claim. By an application of Lemma 68 on the action map $G \times X \to X$ the it is enough to prove that

$$(\star) \qquad \forall x \in X \ \ \forall^* g \in G \ \ \exists V \subseteq_1 G \ \ \forall^* g' \in V \ \ f(g'gx) \in Wf(gx).$$

Let W' be a symmetric open neighborhood of the identity of 1 in H so that $W'W' \subseteq W$ and let (h_n) be a sequence in H so that $(W'h_n)$ covers H. Notice that $Y_n := W'h_n f(x)$ is an analytic subset of Y and therefore $f^{-1}(Y_n)$ has the Baire-property. Since the map $g \mapsto gx$ is continuous we can find a sequence of open subsets O_n of G whose union is dense in G and D_n a comeager subset of O_n so that for all $g \in D_n$ we have that $f(gx) \in W'h_n f(x)$.

The set $D = \cup_n D_n$ is clearly comeager in G. Notice moreover that if $g \in D$ then we can find an open neighborhood V of 1 in G so that $Vg \subset O_n$ for some n. But then $(D_n g^{-1}) \cap V$ is comeager in V and for every g' therein we have that

$$f(g'gx) \in W'h_n f(x) \quad \textit{and} \quad \in f(x) \in W'h_n f(x)$$

Which implies that $f(g'gx) \in W'W'f(gx) \subseteq Wf(gx)$, i.e., $(\star)$ holds. □

Let now $(W_k), (V_n)$ be countable basis of open neighborhoods of $1_H, 1_G$ in H and G respectively. By the above claim there exists a dense G_δ subset C_0 of X so that the function $N \colon C_0 \times \mathbb{N} \to \mathbb{N}$ which assigns to each (x, k) the smallest n so that $\forall^* g \in V_n \ f(gx) \in W_k f(x)$, is well defined. This map is also analytic and therefore by Lemma 67 we can find a dense G_δ set $C_1 \subseteq C_0$ so that $N \restriction C_1 \times \mathbb{N}$ is continuous.

It is easy to see now that C_1 satisfies property (3)(i) above. As we discussed in the beginning of the proof, setting $C = C_1 \cap C_1^{\forall^*}$ concludes the proof. Indeed the reader may check that (3)(ii) is also satisfied. □

9. Week 9

Theorem 65 will be a special case of the following lemma. We first need a definition. Let X be a Polish G-space. The **right Becker digraph** $\mathcal{B}(X/G)$ associated to X is the directed graph whose domain is the quotient X/G whose arrows are precisely all pairs $[x] \to [y]$ so that x right-embeds in y, i.e., so that there is a right-Cauchy sequence (g_n) in G with $g_n y \to x$. Notice that if (g_n) is right Cauchy then $(hg_n g)$ is right Cauchy for every $g, h \in G$. As a consequence if x right-embeds in y then x' right-embeds in y' as well and the definition of the Becker digraph makes sense. If C is any subset of X we denote by $\mathcal{B}(C/G)$ the restriction of $\mathcal{B}(X/G)$ to its subgraph whose domain is $\{[x] \colon x \in C\}$. If $f \colon X \to Y$ is an (E, F)-homomorphism then f induces a map $[f] \colon X/E \to Y/F$ on the quotients. Notice that the map $[f]$ is injective if and only if f is a reduction.

Lemma 70. *Let X be a Polish G-space, Y be a Polish H-space, and let $f \colon X \to Y$ be a Baire-measurable (E^G_X, E^H_Y)-homomorphism. Then there is a comeager subset C of X so that $[f]\restriction(C/G)$ is a digraph homomorphism from $\mathcal{B}(C/G)$ to $\mathcal{B}(Y/H)$.*

Proof. Let C be the comeager subset provided by Lemma 66. If $x, y \in C$ and (g_n) be a right-Cauchy sequence in G with $g_n y \to x$, we will find a right-Cauchy sequence (h_n) in H with $h_n f(y) \to f(x)$.

Let (U^Y_n) be a sequence of open neighborhoods of $f(x)$ in Y so that $\cap_n U^Y_n = \{f(x)\}$ and let (W_n) a sequence of open neighborhoods of 1 in H so that $\cap_n W_n = \{1\}$. We will define (h_n) inductively so that:

(1) $h_n h_m^{-1} \in W_n$ for all $n \geq m$; and
(2) $h_n f(y) \in U_n$ for all n.

As a consequence (h_n) will indeed be right-Cauchy and $h_n f(y) \to f(x)$. In the process of defining (h_n) we will together define a right Cauchy sequence (g'_n) in G which will be Cauchy equivalent to (g_n).

Defining h_0 and g'_0. By (1) and (3)(i) of Lemma 66 we can find an open neighborhood U^X_0 of y in C and an open neighborhood V_0 of 1 in G so that

(1') $f(gz) \in W_0 f(x)$, for all $z \in U^X_0$ and for comeager many $g \in V_0$; and
(2') $f(z) \in U^Y_0$, for all $z \in U^X_0$.

Since (g_n) is right-Cauchy and $g_n y \to x$ we can find k_0 large enough so that for all $k \geq k_0$ we have that $g_k y \in U^X_0$ and $g_{k_0} g_k^{-1} \in \sqrt[3]{V_0}$; see footnote[1]. Since y is in C we find by (2) of Lemma 66 an element $\varepsilon_0 \in G$ very close to 1 ($\varepsilon_0 \in \sqrt[3]{V_0}$ in particular) so that $\varepsilon_0 g_{k_0} y \in U^X_0$ and that moreover $\varepsilon_0 g_{k_0} y \in C$. We set $g'_0 = \varepsilon_0 g_{k_0}$ and let h_0 be any element of H so that $h_0 f(y) = f(g'_0 y)$. Since $g'_0 y \in U^X_0$ we have (2) above for $n = 0$.

It will be instructive to do one more step and leave the general n for the reader.

Defining h_1 and g'_1.

[1]Notation for any symmetric open neighborhood of 1 with $\sqrt[3]{V_0}\sqrt[3]{V_0}\sqrt[3]{V_0} \subseteq V_0$

By (1) and (3)(i) of Lemma 66 we can find an open neighborhood U_1^X of y in C and an open neighborhood $V_1 \subseteq V_0$ of 1 in G so that

(1") $f(gz) \in W_1 f(x)$, for all $z \in U_1^X$ and for comeager many $g \in V_1$; and
(2") $f(z) \in U_1^Y$, for all $z \in U_1^X$.

Again we can chose $k_1 > k_0$, large enough so that for all $k \geq k_1$ we have that $g_{k_1} y \in U_1^X$ and $g_{k_1} g_k^{-1} \in \sqrt[3]{V_1}$.

We want to be able to reach from g'_0 to g_{k_1} with a step within V_0 in order to use (1') above. But notice that $g_{k_1}(g'_0)^{-1} = (g_{k_1}(g_{k_0})^{-1})\varepsilon_0^{-1} \in \sqrt[3]{V_0}\sqrt[3]{V_0} \subseteq V_0$, since $\sqrt{V_0}$ is symmetric. Moreover by (3)(ii) of Lemma 66 we find an element $\varepsilon_1 \in G$ very close to 1 ($\varepsilon_1 \in \sqrt[3]{V_1}$ in particular) so that still we have $\varepsilon_1(g_{k_1}(g'_0)^{-1})(g'_0 y) \in U_1^X$ but moreover the result is also in C.

We set $g'_1 = \varepsilon_1 g_{k_1}$. Since $g'_1(g'_0)^{-1} = \varepsilon_1 g_{k_1}(g'_0)^{-1} = \varepsilon_1(g_{k_1} g_{k_0}^{-1})\varepsilon_0 \in \sqrt[3]{V_0}\sqrt[3]{V_0}\sqrt[3]{V_0} \subseteq V_0$, we can find by (1') above a $w_0 \in W_0$ so that $w_0 h_0 f(y) = f(g'_1 y)$. Set $h_1 = w_0 h_0$. Since $h_1 h_0^{-1} = w_0 \in W_0$ we have (1) above for and since $g'_1 y \in U_1^X$ we have (2) above.

Defining h_n and g'_n. Left to the reader.

□

We can now proceed to the proof of Theorem 65.

Proof of Theorem 65. Let x, y as in the assumption when C is provided by Lemma 70. By Lemma 70 we have that $f(x)$ right embeds in $f(y)$ but since H is CLI this can only happen if $[f(x)] = [f(y)]$. But this contradicts the two assumptions that f is a reduction and that $[x] \neq [y]$. □

Example 71. Let S be any perfect Polish space and consider the Polish S_∞ space $X = S^\omega$ where S_∞ acts by permuting coordinates, i.e.,

$$g(s_0, s_1, \ldots, s_n, \ldots) = (s_{(g_{-1}(0))}, s_{(g_{-1}(1))}, \ldots, s_{(g_{-1}(n))}, \ldots).$$

The induced equivalence relation $E_X^{S_\infty}$ is called countable sets of reals and it is often denoted by $=^+$. Indeed notice that

$$(s_0, s_1, \ldots, s_n, \ldots) E_X^{S_\infty} (t_0, t_1, \ldots, t_n, \ldots) \iff$$

$$\iff \{s_0, s_1, \ldots, s_n, \ldots\} = \{t_0, t_1, \ldots, t_n, \ldots\}$$

If (g_m) is right-Cauchy sequence then (g_m^{-1}) is left Cauchy and it converges to some embedding from ω to ω. In other words,

$$g_m(s_0, s_1, \ldots, s_n, \ldots) \to_{m\to\infty} (t_0, t_1, \ldots, t_n, \ldots)$$

if and only if $s_{\gamma(n)} = t_n$ for all n. So if we restrict our attention to the dense G_δ subset Inj of X of all injective sequences we have that

$$(t_0, t_1, \ldots, t_n, \ldots) \text{ right-embeds in } (s_0, s_1, \ldots, s_n, \ldots) \iff$$

$$\iff \{t_0, t_1, \ldots, t_n, \ldots\} \subseteq \{s_0, s_1, \ldots, s_n, \ldots\}.$$

Theorem 72. *The S_∞-space X from Example 71 satisfies the conditions of Theorem 65 and therefore $E_X^{S_\infty}$ it is not classifiable by CLI-group actions.*

Proof. Let $\sigma\colon S^\omega \to S^\omega$ be the unilateral left shift, i.e., $\sigma\colon (s_0, s_1, \ldots, s_n, \ldots) = (s_1, s_2, \ldots, s_{n+1}, \ldots)$ and notice that σ restricts to a continuous open surjective map from Inj to Inj. If C is any comeager subset of Inj the set $\sigma^{-1}(C)$ is also comeager. Let $\bar{t} \in C \cap \sigma^{-1}(C)$. Then there is $\bar{s} \in C$ with $\sigma\bar{s} = \bar{t}$ and as a consequence $\{t_0, t_1, \ldots, t_n, \ldots\} \subseteq \{s_0, s_1, \ldots, s_n, \ldots\}$. So, $\bar{t}$ right embeds in $\bar{s}$. □

We can now deduce Theorem 62 as a corollary of the above Theorem.

Proof of Theorem 62. Let S be the unit circle in the complex plane and fix an orthonormal basis (e_n) of $\mathcal{H}$. Sending each element $(s_0, s_1, \ldots, s_n, \ldots)$ of $X = S^{omega}$ to the diagonal operator T with $T(e_n) = s_n e_n$ gives as a reduction from $(S^\omega, E_X^{S_\infty})$ to $(\mathcal{U}(\mathcal{H}), \simeq_U)$

□

Proof of Theorem 63. Assume that the G-space X is turbulent and that E_X^G Borel reduces via f to the orbit equivalence relation $E_Y^{S_\infty}$ on the S_∞-space Y.

Let C be the comeager subset provided by Lemma 66. Since orbits in X are meager we can find $x, x' \in C$ lying in different turbulent orbits, i.e., $[x] \neq [x']$ and every $y \in [x] \cap [x']$ satisfies the assumptions (1),(3) from Theorem 63.

We claim that $f(x)$ and $f(x')$ are in the same orbit, contradicting the fact that f is a reduction. In order to prove our claim we fix complete metrics d and ρ on S_∞ and Y respectively and we inductively produce:

$$U_0,\ W_0,\ h_0',\ |\ U_0',\ W_0',\ h_0,\ |\ U_1,\ W_1,\ h_1',\ |\ U_1',\ W_1',\ h_1, \dots$$

So that U_n and U_n' are open subsets of Y with $\mathrm{diam}_\rho(U_n), \mathrm{diam}_\rho(U_n') \to 0$ and

$$U_0 \supseteq \mathrm{cl}(U_0') \supseteq U_0' \supseteq \mathrm{cl}(U_1) \supseteq U_1 \supseteq \mathrm{cl}(U_1') \supseteq \dots,$$

(h_n) and (h_n') are sequences in G and W_n, W_n' are open subgroups of S_∞ so that

(1) $h_n \cdots h_0 f(x) \in U_n'$ and $h_n' \cdots h_0' f(x') \in U_n$;
(2) $h_n \in W_n$ and $h_n' \in W_{n-1}'$, with $W_{-1} = S_\infty$;
(3) for every $n > 0$ we have that

$$W_n \subseteq \{h \in S_\infty \mid d(hh_{n-1} \cdots h_0, h_{n-1} \cdots h_0) < 1/2^n\},$$
$$W_n' \subseteq \{h \in S_\infty \mid d(hh_n' \cdots h_0', h_n' \cdots h_0') < 1/2^n\}.$$

Assuming that we have constructed all the data above notice that $h_n \cdots h_0 x$ converges to the same place that $h_n' \cdots h_0' x'$ converges, call it $y \in Y$. Moreover $(h_n \cdots h_0)$ and $h_n' \cdots h_0'$ are both d-Cauchy sequences and therefore they converge to some $h, h' \in S_\infty$. As a consequence $hx = y = h'x'$ and therefore indeed $[x] = [x']$.

We describe the first steps in the construction and leave the induction for the reader. At each stage we will use (1) and (3) of Lemma 66 and the assumption that $\mathrm{int}(\overline{\mathcal{O}(z, U, V)}) \neq \emptyset$ to produce h_n and h_n'. We will assume below that:

($\star$) Properties (2) and (3) of Lemma 66 hold **for all** g rather than for comeager many g in G and in V respectively.

This simplifying assumption we will simplify our exposition but as in the proof of Theorem 65 we can always modify this "approximate argument" to an actual argument.

Step 0(remember the assumption $\star$). Let U_0 be any open neighborhood of $f(x)$ with $\mathrm{diam}_\rho(U_0) < 1$ and let W_0 be any open subgroup of S_∞. By Lemma (1) and (3) of Lemma 66 we can find an open neighborhood $\widehat{U_0}$ of x in X and an open neighborhood V_0 of 1 in G so that for all $z \in \widehat{U_0}$ and every $g \in V_0$ there is $h \in W_0$ so that $f(gz) = hf(z)$ and $f(z) \in U_0$. Moreover by assumption the local orbit $\mathcal{O}(x, \widehat{U_0}, V_0))$ is dense in some non-empty open set $O_0 \subseteq \widehat{U_0}$. Since $[x']$ is dense in X we can find $g_0' \in G$ so that $g_0' x' \in O_0$. Since f is a reduction, there is $h \in S_\infty$ so that $hf(x') = f(g_0' x')$. Set h_0' to be this h.

Step $0'$(remember the assumption $\star$). Let U_0' be any open neighborhood of $h_0' f(x')$ with $\mathrm{cl}(U_0') \subseteq U_0$ and let W_0' be any open subgroup of S_∞ with $W_0' \subseteq \{h \in S_\infty \mid$

$d(hh'_0, h'_0) < 1\}$. By Lemma (1) and (3) of Lemma 66 we can find an open neighborhood $\widehat{U'_0}$ of g'_0x in X and an open neighborhood V'_0 of 1 in G so that for all $z \in \widehat{U'_0}$ and every $g \in V'_0$ there is $h \in W'_0$ so that $f(gz) = hf(z)$ and $f(z) \in U'_0$. We can assume without loss of generality that $\widehat{U'_0} \subseteq O_0$. Moreover by assumption the local orbit $\mathcal{O}(g'_0x', \widehat{U'_0}, V'_0))$ is dense in some non-empty open set $O'_0 \subseteq \widehat{U'_0}$.

Since O'_0 is a subset of $\mathcal{O}(x, \widehat{U_0}, V_0))$ we can find a finite sequence $g_0^k, \ldots, g_0^1, g_0^0 \in V_0$ so that $g_0^k \cdots g_0^1 g_0^0 x \in O'_0$ and $g_0^i \cdots g_0^1 g_0^0 x \in \widehat{U_0}$ for every $i < k$. As a consequence, by definition of $\widehat{U_0}, V_0$ we can find $h_0^k, \ldots, h_0^1, h_0^0 \in W_0$ so that $h_0^k \cdots h_0^1 h_0^0 f(x) \in U'_0$. But since W_0 is a group, we get a single $h_0 := h_0^k \cdots h_0^1 h_0^0 \in W_0$ with $h_0 f(x) \in U'_0$

Steps n, n' are similar to $0'$ and are left to the reader.

□

We will now give some applications. The first example shows that the classification of all asymptotic behaviors of real valued sequences up to certain standard notions of equivalence is not classifiable by countable structures.

Example 73. Let G be the Banach space $c_0(\omega)$ of all sequences of reals converging to 0 with the supremum norm, or any of the $l_p(\omega)$ spaces where $0 \in [1, \infty)$. We view G as a Polish group with addition which acts on the space $\mathbb{R}^\omega$ by translation

$$(g_0, g_1, \ldots) \cdot (x_0, x_1, \ldots) := (g_0 + x_0, g_1 + x_1, \ldots).$$

The space $\mathbb{R}^\omega$ is endowed with the product topology.

Theorem 74. *The action of G on $\mathbb{R}^\omega$ described in the above example is turbulent.*

Proof. It is easy to see that the orbits of G on $\mathbb{R}^\omega$ are meager. We show here that $\mathcal{O}(\bar{x}, U, V))$ is somewhere dense, for every $\bar{x} \in \mathbb{R}^\omega$, every open $U \subseteq \mathbb{R}^\omega$ containing $\bar{x}$ and every open neighborhood of 1 in G.

For simplicity we can assume that $\bar{x} = \bar{0} = (0, 0, 0, \ldots)$. By shrinking U if necessary we can assume that it is of the form

$$\{\bar{x} \in \mathrm{R}^\omega \mid |x_0| < \delta, \ldots, |x_n| < \delta\},$$

for some $\delta > 0$ and $n \in \mathbb{N}$. Let also $V = \{\bar{g} \ inG \mid ||\bar{g}|| < \varepsilon\}$. We claim that $\mathcal{O}(\bar{x}, U, V))$ is dense in U. For that let $U' \subseteq U$. We can assume that

$$U' = \{\bar{x} \in \mathrm{R}^\omega \mid |x_0 - a_0| < \delta', \ldots, |x_n - a_n| < \delta', \ldots, |x_N - a_N| < \delta'\},$$

for some $N \geq n$ and some sequence $(a_0, \ldots, a_N) \in \mathbb{R}^{N+1}$ with $|a_i| < \delta$ when $i \leq n$.

Notice that there exists some $M > 0$ large enough so that $\bar{g} = (a_0/M, \ldots, a_N/M, \ldots) \in V$ and by applying $\bar{g}$ M-many times on $\bar{0}$ we hit U' while staying in the process entirely within U. □

Next we prove Theorem 59. For that we will first recall some standard notions from functional analysis and prove a lemma for turbulent actions. We start with the lemma.

Lemma 75. *Let X be a Polish G-space. Then X is turbulent if every orbit is meager, there is some dense orbit, and there exists some $x \in X$ that is turbulent, i.e., for every $U \subseteq_x X$ open and every $V \subseteq_1 G$ open we have that $\mathcal{O}(x, U, V)$ is somewhere dense.*

Proof. We will show that there is a comeager set $C \subseteq X$ so that every $z \in C$ is turbulent.

Claim. *(i) every $y \in [x]$ is turbulent; (ii) the orbit $[x]$ is dense.*

Proof of Claim. (i) Let $g \in G$ with $gx = y$ and let $U \subseteq_y X$ open and $V \subseteq_1 G$ open. Since x is turbulent then we can find a non-empty open $O \subseteq U$ in which $\mathcal{O}(x, g^{-1}U, gVg^{-1})$ is dense. We claim that $\mathcal{O}(y, U, V)$ is dense in gO. To see this let gO' be a non-empty open subset of gO and let $g_n, \ldots, g_0$ be the witness forthe $(g^{-1}U, gVg^{-1})$-path with $g_n \cdots g_0 x \in O'$ and notice that $g^{-1}g_n g, \ldots, g^{-1}g_0 g$ is a (U, V)-path landing in gO'.

(ii) let O be a nonempty open subset of X and let $[z]$ be some dense orbit. If $\mathcal{O}(x, U, V)$ is any (U, V)-local orbit then by density of $[z]$ we can assume that $z \in \mathcal{O}(x, U, V)$ and $gz \in O$ for some $g \in G$. But then if $g_n, \ldots, g_0$ is the witness for a (U, V)-path with $g_n \cdots g_0 x \in g^{-1}O$ we have that $gg_n \cdots g_0 x \in O$. Since O is arbitrary, $[x]$ is dense. □

Let $\mathcal{Q}$ be a countable dense subset of G, let $\mathcal{U}$ be a countable basis for the topology of X consisting of open sets, and let $\mathcal{V}$ be a countable basis of open neighborhoods of 1 in G.

For every $g \in \mathcal{Q}$, $U \in \mathcal{U}$ and $V \in \mathcal{V}$ with $gx \in U$ let $O_{g,U,V}$ be an non-empty open subset of X, in which $\mathcal{O}(gx, U, V)$ is dense. This exists by (i) of the above claim. Let $B_{U,V}$ be the collection of all $y \in X$ so that either $y \notin U$ or there is some $g \in \mathcal{Q}$ such that

$$\mathcal{O}(y, U, V) \cap O_{g,U,V} \neq \emptyset.$$

Notice that $B_{U,V}$ is G_δ by definition, and dense by (ii) of the above claim, since it contains $[x]$.

For every g, U, V as above, and every $U' \in \mathcal{U}$ with $U' \cap O_{g,U,V} \neq \emptyset$ let $C_{g,U,V,U'}$ be the collection of all $y \in O_{g,U,V}$ so that

$$\mathcal{O}(y, U, V) \cap U' \neq \emptyset.$$

Notice that $C_{g,U,V,U'}$ is open and dense in $O_{g,U,V}$. As a consequence, if we set $D_{g,U,V}$ to be the set of all y so that

$$\forall U' \in \mathcal{U} \text{ with } U' \cap O_{g,U,V} \neq \emptyset \;\; \big(y \in O_{g,U,V} \implies \mathcal{O}(y, U, V) \cap U' \neq \emptyset\big)$$

then $D_{g,U,V}$ is a dense G_δ. But then if we set

$$C = \bigcap_{h,g\in Q;\, U\in\mathcal{U},\, gx\in U;\, V\in\mathcal{V}} hB_{U,V} \cap hD_{g,U,V},$$

we have that C is dense G_δ and moreover:

Check. For every $y \in C$ we have that y is turbulent. □

We recall now some basic facts for unitary operators. First recall that if $T \in \mathcal{U}(\mathcal{H})$ then the spectrum $\sigma(T)$ is the collection of all $\lambda \in \mathbb{T} \subseteq \mathbb{C}$ so that $T - \lambda I$ is not invertible. Contrary to the case when $\dim(\mathcal{H}) < \infty$, such λ is not an eigenvalue, i.e., there does not always exist $v \in \mathcal{H}$ with $Tv = \lambda v$. However each $\lambda \in \sigma(T)$ can be shown to be an *approximate eigenvalue*, i.e., there is a sequence v_n with $||v_n|| = 1$ so that $(T - \lambda I)v_n \to 0$.

Example 76. Let $\mathcal{H} = l_2(\mathbb{Z})$ and T be the right shift given by $(Tf)(n) = f(n-1)$. Notice that if $\lambda \in \mathbb{T}$ then there is no $f \in l_2(\mathbb{Z})$ with $Tf = \lambda f$. However one can find an element $\phi \in l_\infty(\mathbb{Z})$ with $T\phi = \lambda\phi$, namely

$$\phi = (\ldots, \lambda^2, \lambda, 1, \lambda^{-1}, \lambda^{-2}, \ldots).$$

By truncating and normalizing we get a sequence

$$f_n = \frac{1}{\sqrt{2n+1}}(\ldots, 0, 0, \lambda^n \ldots, \lambda, 1, \lambda^{-1}, \ldots \lambda^{-n}, 0, 0, \ldots)$$

in $l_2(\mathbb{Z})$ which witnesses that λ is an approximate eigenvalue.

Similarly when $\dim(\mathcal{H})$ is infinite eigenspaces are replaced with spectral measures. A **spectral measure** is a map $E : \mathcal{B}(\mathbb{T}) \to \mathcal{P}(\mathcal{H})$ from the Borel σ-algebra of $\mathbb{T}$ to the space of all projections on $\mathcal{H}$ (all bounded linear operators with $P = P^*$ and $P^2 = P$) satisfying the following properties:

(1) $E(\mathbb{T}) = 1$
(2) if (B_n) is a disjoint family of Borel sets then $E(\bigcup_n B_n) = \Sigma_n E(B_n)$

Notice that for every Borel subset B of $\mathbb{T}$ and $v, w \in \mathcal{H}$ we have that $\langle E(B)v, w\rangle$ is a complex number. Hence for fixed $v, w \in \mathcal{H}$ the integral

$$\int_{\mathbb{T}} \lambda d\langle E_\lambda v, w\rangle$$

In fact we have the following theorem which we will use without a proof.

Theorem 77 (Spectral theorem for $\mathcal{U}(\mathcal{H})$)**.** *For every $T \in \mathcal{U}(\mathcal{H})$ there exists a spectral measure E^T so that for all $v, w \in \mathcal{H}$ we have that*

$$\langle Tv, w\rangle = \int_{\mathbb{T}} \lambda d\langle E^T_\lambda v, w\rangle$$

If we fix now an orthonormal basis (e_n) we have an assignment $T \mapsto \mu_T$ from $\mathcal{U}(\mathcal{H})$ to the space $\mathcal{P}(\mathbb{T})$ of probability measures on $\mathbb{T}$, given by

$$\mu_T(B) = \sum_n \frac{1}{2^n} \int_{\mathbb{T}} \lambda d\langle E^T_\lambda e_n, e_n\rangle$$

This assignment depends on the chosen basis (e_n) but only up to measure equivalence. The measure μ_T is called the (representative corresponding to (e_n) of) **maximal spectral type of** T. One fact we are going to use is that the assignment $T \mapsto \mu_T$ is continuous when $\mathcal{P}(\mathbb{T})$ is endowed with the weak convergence topology ($\mu_n \to \mu$ if and only if for every continuous $f\colon \mathbb{T} \to \mathbb{R}$ we have that $\int f d\mu_n \to \int f d\mu$)

We can now proceed to the proof of Theorem 59.

Proof of Theorem 59. Let $g, T \in \mathcal{U}(\mathcal{H})$ we denote by $g \cdot T$ the element gTg^{-1} resulting from the action of g on T by conjugation.

Claim. *If $T \in \mathcal{U}(\mathcal{H})$ with $\sigma(T) = \mathbb{T}$ then T with has a dense conjugacy class $[T]$ in $\mathcal{U}(\mathcal{H})$*

Proof of Claim. First notice the set $D \subseteq \mathcal{U}(\mathcal{H})$ of operators with discrete spectrum of finite multiplicity is dense in $\mathcal{U}(\mathcal{H})$. To see this Let U be any open subset of $\mathcal{U}(\mathcal{H})$. By shrinking we can assume that U is of the form

$$U_{R,N,\delta} = \{S \in \mathcal{U}(\mathcal{H}) \mid |\langle Se_k, e_l\rangle - \langle Re_k, e_l\rangle| < \delta, k, l \leq N\}$$

for some $R \in \mathcal{U}(\mathcal{H})$, $N \in \mathbb{N}$, and $\delta > 0$. Let A be any unitary operator which is equal to identity outside of the finite dimensional Hilbert subspace $\mathcal{H}'$ spanned by $\{e_1, \ldots, e_N\} \cup \{R(e_1), \ldots, R(e_N)\}$ and so that $A(e_1) = R(e_1), \ldots, A(e_N) = R(e_N)$. Then $A \in D \cap U$

But any $A \in D$ can be approximated by conjugates of T. To see this let $v_1, \ldots, v_n$ be finitely many the eigenvectors of A with eigenvalues $\lambda_1, \ldots, \lambda_n$. By ε-pertubing we can assume that $\lambda_i \neq \lambda_j$ and hence $v_i \perp v_j$. Since T is of full spectrum we can find $w_1, \ldots, w_n$ ε-approximate eigenvectors of T with eigenvalues $\lambda_1, \ldots, \lambda_n$ which are orthogonal. But then any change of basis sending v_i to w_i will conjugate T to A. □

Claim. *For every $T \in \mathcal{U}(\mathcal{H})$ the orbit $[T]$ of T is meager.*

Proof of Claim. First we point out that the collection $U_C \subseteq \mathcal{U}(\mathcal{H})$ of all operators with continuous spectrum (no atoms) is an invariant dense G_δ subset of $\mathcal{U}(\mathcal{H})$. Density follows by the previous claim and the fact that the right shift on $l_2(\mathbb{Z})$ is in U_C; see example above. It is G_δ since it is the pre-image under the continuous assignment $T \mapsto \mu_T$ of the G_δ subset of $\mathcal{P}(\mathbb{T})$ of all atomless probability measures (add argument).

So it suffice to show that orbits in U_C are meager. Let $T \in U_C$ and let Let μ be the maximal spectral measure of T. Consider the set $L_T = \{S \in \mathcal{U}(\mathcal{H}) \mid \mu_S \perp \mu\}$ be the collection of all operators whose maximal spectral measure μ_S and μ are mutually singular. It suffice to show that L_T is dense G_δ in U_T. But L_T is dense since we can always find a dense subset D of $\mathbb{T}$ with μ-measure 0 and then construct an $S \in U_T$ with $\mu_S(D) = 1$ (add argument). Moreover L_T is G_δ since it is the preimage under $S \mapsto \mu_S$ of measures ν so that for all $\varepsilon > 0$ there is an open set $U \subseteq \mathbb{T}$ so that $\mu(U) < \varepsilon$ and $\nu(U) > 1 - \varepsilon$. Notice that the set $\{\nu \in U_C \mid \nu(U) > 1 - \varepsilon\}$ is open for every U, ε since ν varies over U_C.

□

Let (e_j) be an orthonormal basis of $\mathcal{H}$ and $(e_{i,j})_{i,j \in \mathbb{N}\times\mathbb{N}}$ be some re-indexing of (e_i). Let (T_i) be a countable and dense sequence in $\mathcal{U}(\mathcal{H})$ so that T_i is the identity operator in a subspace of $\mathcal{H}$ spanned by all but many $e_i's$.

We define T in $\mathcal{U}(\mathcal{H})$ by copying, for every fixed i, the action of T_i on $(e_j)_j$ to an action of T on $(e_{i,j})_j$, i.e.,

$$T(e_{i,j}) = \Sigma_n \lambda_n e_{i,n} \iff T_i(e_j) = \Sigma_n \lambda_n e_n.$$

Claim. *T is turbulent.*

Proof of Claim. Let $U \subseteq_T \mathcal{U}(\mathcal{H})$ and let $V \subseteq_1 \mathcal{U}(\mathcal{H})$ be open sets. By shrinking these sets further we can assume that

$$U = \{S \in \mathcal{U}(\mathcal{H}) \mid \forall m, n \leq K \; |\langle S(e_m), e_n\rangle - \langle T(e_m), e_n\rangle| < \delta\},$$

$$V = \{g \in \mathcal{U}(\mathcal{H}) \mid \forall m \leq K \; |\langle S(e_m), e_m\rangle - 1| < \delta\},$$

for some common $K > 0$ and $\delta > 0$. We will show that $\mathcal{O}(T, U, V)$ is dense in U. For that let O be a non-empty open subset of U. We can assume without loss of generality that there is some $L \geq K$ so that if S, S' satisfy $\langle S(e_m), e_n\rangle = \langle S'(e_m), e_n\rangle$ for all $m, n \leq L$ then $S \in O \iff S' \in O$.

By the choice of $T_i's$ we can find N large enough so that $\langle T(e_l), e_{i,n}\rangle = 0$ for all $n \geq N$ and $l \leq L$. Find m large enough so that $e_{m,j} \in \{e_n : n \geq N\}$ and that moreover $T_m \in O$.

Set $g_{\pi/2}(e_{m,j}) = e_j$ and $g_{\pi/2}(e_j) = -e_{m,j}$ for all $i \leq L$ and $g_{\pi/2}(e_j) = e_j$ otherwise. It is easy to see that $g_{\pi/2} T g_{\pi/2}^{-1} = T_m \in O$. We will witness this "jump" by a (U, V)-path.

For every $\theta \in [0, \pi/2]$ consider the operator g_θ defined by

$$g_\theta(e_{m,j}) = \sin(\theta) e_j + \cos(\theta) e_{m,j},$$

$$g_\theta(e_j) = -\sin(\theta) e_{m,j} + \cos(\theta) e_j,$$

if $j \leq L$ and $g_\theta(e_i) = e_j$ otherwise.

Notice that for large enough M we have that $g_{\pi/2M} \in V$. The rest is just checking.

□

□

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Mathematics Department, Caltech, 1200 E. California Blvd, Pasadena, CA 91125
Email address: `panagio@caltech.edu`
URL: `http://www.its.caltech.edu/~panagio/`